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A modified CKKW matrix element merging approach to angular-ordered parton showers

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Abstract: A modified version of the CKKW matrix element merging algorithm is presented, suitable for use in an angular-ordered parton shower, using truncated showers and forced splittings. The algorithm is implemented in the Herwig++ Monte Carlo event generator for the benchmark process $e^+e^- \to$ hadrons. Results are presented at parton and hadron levels, demonstrating a smooth merging between the matrix elements and parton shower and an improved description of LEP data.

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Contents

1 Introduction

Monte Carlo event generators have been successful in providing a full simulation of the physics at collider experiments and have proven to be invaluable tools in both planning future experiments and analysing data from current experiments. They use a flexible and convenient event-by-event description which allows a range of physics models to be implemented and as such they describe a wide variety of phenomena. In particular, they provide a means, via the parton shower, of evolving from hard scales, where partons are produced in fixed-order perturbation theory, to soft scales where non-perturbative models must be applied.

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Besides the simple fixed-order matrix elements which describe the hard processes, the key perturbative element of Monte Carlo event generators is the parton shower. In this phase of the simulation, partons undergo DGLAP evolution from some initial scale, characteristic of the underlying hard scattering, down to the hadronization scale. In Herwig++[1, 2] the evolution variable of the parton shower is chosen such that branchings are ordered in their opening angles, accounting for the effects of QCD coherence [3]. This corresponds to resumming next-to-leading logarithm (NLL) terms to all orders in perturbation theory.

While the parton shower accurately simulates soft and collinear radiation, it does not provide a reliable description of hard (high transverse momentum) emissions. In fact, requiring that the regions of phase space into which each *shower progenitor* (the partons which initiate the shower) can emit do not overlap, to avoid double counting, gives rise to a *dead zone*: a region of phase space corresponding to high transverse momentum, wide angle, emissions which none of the showering particles can emit into.[1] Even within the accessible *shower regions* of phase space, the distribution of radiation involves some degree of approximation, since at any given fixed order in perturbation theory, the parton shower effectively approximates the real emission corrections to the hard scattering process by a product of splitting functions and Sudakov form factors, summed over all combinations of branchings which give rise to the same final state. These approximations account for the NLL corrections associated with soft and collinear radiation in the perturbative series.

In order to improve the description in the large transverse momentum region, the parton shower can be combined with exact fixed-order matrix elements. The earliest and simplest means of forming this combination is known as the matrix element correction method [4]. This corrects the hardest emission generated by the parton shower such that it is distributed according to the real single emission matrix element squared. This technique has been successfully applied to important processes in a number of generators [5, 6], including Herwig++ [2].

In recent years more general matrix element-parton shower merging algorithms have been introduced. These combine tree-level matrix elements with parton showers, for a given process, for all parton multiplicities below some maximum N. Hence these algorithms correct all distributions involving up to N external partons, instead of just that of the hardest emission. Several schemes of this type have been developed and successfully implemented in event generators. The most well known of these are the CKKW [7–9], CKKW-L [10], MLM [11] and pseudo-shower [12] methods. All these methods have the same general approach [13, 14] whereby the phase space for parton emissions is divided into two regions by a merging scale y_{MS}, defined in some jet measure. Above the merging scale, emissions are described by exact matrix elements while below it emissions are produced by the parton shower.

[1] This is a feature of parton showers such as Herwig++ , which describe $1 \to 2$-parton branchings and where it is not possible to choose the initial conditions of the shower lines in order to achieve a complete coverage of the emission phase-space, without introducing double-counting from the overlap of the emission phase space of other shower lines. Dipole showers, describing $2 \to 3$-parton branchings are able to cover the full emission phase space and therefore to not have this problem.

In this article we present a matrix element merging scheme based on the CKKW algorithm. A fundamental ingredient in the CKKW method is the association of a *pseudo-shower history* to the configurations generated according to the fixed-order matrix elements. Each shower history is constructed by clustering the two most closely separated partons, according to the transverse momentum measure defining the merging scale, until a leading-order parton configuration is obtained. The resulting branchings in the shower history are therefore ordered according to the jet measure, which may not equate to the ordering variable of the parton shower, as is the case for the angular-ordered parton shower of Herwig++. This discrepancy is understood to give rise to serious problems, in particular it spoils the colour coherence properties of these shower algorithms. Although this was already noted, and an attempt made to address it, in the original CKKW paper, realisations of the method highlight the fact that the colour structure in the events is nevertheless in conflict with that expected on the grounds of colour coherence, moreover, they show that this is not simply an esoteric consideration but a cause of significant practical problems, including a dependence on the unphysical merging scale [12, 13].

We shall present and validate a modified version of the CKKW method, intended to optimise the implementation of these colour coherence effects, by a fully consistent merging of an angular-ordered parton shower with fixed-order matrix elements. The idea behind this method was originally proposed by Nason in ref. [15]. The central result of that theoretical work is the observation that the parton shower may be formally decomposed in terms of *truncated showers*, *hard emissions*, and *vetoed showers*. Reference [15] advocates that the CKKW algorithm may then best model the coherent emission of radiation by including these truncated showers, consisting of only soft emissions, prior to, and between, the hard emissions in the shower history, thereby rendering it angular-ordered. In the following we will develop the full details necessary for our practical implementation of this idea for the process $e^+e^- \to$ hadrons and compare the results of it to LEP data.

This work builds on the infra-structure introduced in refs. [18, 19] for the Herwig++ angular-ordered parton shower, where the truncated shower was initially developed for use in the POWHEG [15–17] next-to-leading order matching scheme applied to Drell-Yan vector boson production. This was the first time a full implementation of the truncated shower had been developed.[2]

While this work was in production, ref. [20] appeared on the arXiv, employing similar techniques to those described in the present work for matrix element merging with a transverse-momentum-ordered dipole shower.

The paper is set out as follows. In section 2 we review the original CKKW merging prescription. In section 3 we go on to describe the way in which the angular-ordered parton shower may be decomposed into hard emissions, truncated showers, and vetoed showers. Having introduced the relevant conceptual ingredients we then give a more detailed technical description of our modified CKKW algorithm in section 4. In section 5 we present a validation of our algorithm by comparing to LEP data for $e^+e^- \to$ hadrons, before giving our conclusions in section 6.

[2] An approximate, single emission truncated shower is also described in refs. [21, 22].

2 CKKW merging

In this section we present an overview and discussion of the original CKKW algorithm for the process $e^+e^- \to$ hadrons. We first describe the algorithm for the case where the parton shower evolution variable is identical to the merging variable before describing the adaptations which must be made for the Herwig++ angular-ordered parton shower.

2.1 Transverse-momentum-ordered CKKW merging

The algorithm is simplest if the merging variable is the same as the ordering variable of the parton shower. We therefore consider first the case where we have a single transverse momentum variable q as the parton shower evolution and merging variables.

The basic principle underlying the CKKW approach, is that the distribution of radiation in the region of phase space where all partons are separated by an amount q greater than the *merging scale* $q_{\rm MS}$, should be given by tree-level matrix elements, while for $q \leq q_{\rm MS}$ it should be given by the parton shower. The algorithm then requires, as input, samples of events of the process with up to N partons in the final state. These input samples are easily obtained using fully automated tree-level event generators such as Madgraph/MadEvent [25, 26].[3] As well as producing the events themselves, for each sample with n partons the generator will provide a finite, tree-level, jet cross section $\sigma_n^{(ME)}(q_{\rm MS})$.

Naïvely, with the input events in hand, one might then consider filling the remaining phase space by selecting events from each sample with n partons, with a probability proportional to $\sigma_n^{(ME)}$, and simply invoking the parton shower on each of the external legs, starting from the scale $q_{\rm MS}$. However, the merging scale $q_{\rm MS}$ is not a physical parameter and so all distributions of partons should be insensitive to its value. This would certainly not be the case for such a naïve procedure, since the distribution of radiation from the parton shower and the fixed-order matrix elements are known to differ, especially in the regions corresponding to high and low q emissions. The great success of the CKKW algorithm is in its ability to correct for the mismatch at the phase space partition $q_{\rm MS}$ by providing a smooth, physical, interpolation between the matrix element distribution at high q values and that of the parton shower in the low q region.

To illustrate how this works, consider the simplified case of merging only samples of 2- and 3-parton events, with $q \geq q_{\rm MS}$, for $e^+e^- \to$ hadrons, with a q-ordered parton shower. In general, the parton shower cross section analogous to $\sigma_n^{(ME)}(q_{\rm MS})$, with n partons resolved at the merging scale, may be written as the product of the leading-order cross section together with a set of Sudakov form factors and splitting functions. The product of these splitting functions and the leading-order cross section approximate the exact tree-level n-jet cross section. For the case of three partons this cross section is

$$\sigma_3^{(PS)}(q_{\rm MS}) = \sigma_2 \times 2\left[\Delta_q(q_I, q_{\rm MS})\right]^2 \int_{q_{\rm MS}}^{q_I} {\rm d}q\, \alpha_S(q)\, \Gamma_{q\to qg}(q)\, \Delta_g(q, q_{\rm MS}), \tag{2.1}$$

where q_I is the scale at which the parton shower is initiated and $\alpha_S(q)\Gamma_{\widetilde{ij}\to ij}(q)$ is the probability for a parent parton $\widetilde{ij}$ to branch into two daughter partons i and j, in the interval

[3] Currently, computational efficiency limits the total number of final-state particles to around six.

$[q, q + dq]$.[4] The overall normalisation factor σ_2 is simply the leading-order cross section. Finally, in eq. (2.1), $\Delta_{\widetilde{ij}}(q, q_{\text{MS}})$ is the Sudakov form factor, which can be interpreted as the probability for the parent parton $\widetilde{ij}$ to evolve from a scale q down to the scale q_{MS} without undergoing a resolvable branching,

$$\Delta_{\widetilde{ij}}(q, q_{\text{MS}}) = \exp\left[- \sum_{\widetilde{ij} \to ij} \int_{q_{\text{MS}}}^{q} \mathrm{d}q' \, \alpha_S \left(q'\right) \Gamma_{\widetilde{ij} \to ij} \left(q'\right) \right]. \tag{2.2}$$

In parton shower (NLL) expansions of the jet cross sections, such as that in eq. (2.1), the exact tree-level matrix elements are approximated by the product of the leading-order cross section and splitting functions. In order to improve the parton shower with exact tree-level matrix elements, this product should be replaced by the corresponding exact, tree-level jet cross section.

The CKKW merging should not affect the NLL expansion of the jet cross section therefore the NLL expansion of the matrix element contribution should give the result in eq. (2.1). Since a NLL expansion of the tree-level matrix elements yields a corresponding product of parton shower splitting functions, it is clear that in order to retain the NLL form of eq. (2.1), the matrix element contribution above q_{MS} should be given by configurations generated according to the tree-level jet cross sections reweighted by appropriate Sudakov and running α_S factors.

In order to determine appropriate reweighting factors for events from the tree-level generator, a pseudo-shower history must be assigned to each event. This shower history interprets the set of external parton momenta as a set of branchings originating from a leading-order configuration. This procedure gives rise to a set of *nodal values*, q_i, for the scales at which each pseudo-branching occurred. These scales provide the arguments for the Sudakov form factors and α_S factors with which the configuration should be reweighted. In the original CKKW publication, this pseudo-shower history is assigned by repeatedly clustering the pair of partons[5] with the smallest separation according to the jet resolution variable, until only the particles of the leading-order process remain.

In the case being considered, where the evolution variable has been taken to match the merging scale variable, combining the matrix elements with the parton shower is straightforward. The parton shower evolution can be split into two parts: first an evolution from the initial scale down to the merging scale q_{MS}; then an evolution from the merging scale down to the hadronization scale q_0. This results in a simple procedure for attaching the parton shower to the reweighted matrix elements, where each external parton produces a shower line evolving from the merging scale.

The full CKKW algorithm then proceeds as follows:

1. a jet multiplicity n is generated with probability

$$P_n = \frac{\sigma_n^{(ME)}(q_{\text{MS}})}{\sum_N \sigma_i^{(ME)}(q_{\text{MS}})}, \tag{2.3}$$

[4] The dependence on auxiliary splitting variables has been suppressed.

[5] Only pairs of partons whose flavours correspond to allowed branchings are considered.

where all cross sections are evaluated at a fixed strong coupling $\alpha_{S_{\rm ME}}$;

2. a configuration of n parton momenta is generated according to $\mathrm{d}\sigma_n^{(ME)}(q_{\rm MS})$;

3. external partons are clustered, defining a pseudo-shower history with a set of nodal scales q_i;

4. the configuration is reweighted by Sudakov and α_S factors: each internal line between two nodes at q_i and q_{i+1} contributes a factor of $\Delta_f(q_i, q_{\rm MS})/\Delta_f(q_{i+1}, q_{\rm MS})$, each external line emanating from a node with scale q_i contributes $\Delta_f(q_i, q_{\rm MS})$, while each node itself contributes $\frac{\alpha_S(q_i)}{\alpha_{S_{\rm ME}}}$;

5. the parton shower is invoked on each external parton from a starting scale of $q_{\rm MS}$.

This scheme is independent of the merging scale to NLL order [7]. We have reweighted configurations such that the NLL three-jet cross section resolved at the merging scale is given by eq. (2.1). This NLL cancellation of merging scale dependence can be seen by considering the cross section for three jets resolved at the hadronization scale. This cross section is given by the sum of the probability of generating a single emission in the matrix element region and none in the parton shower, together with the probability of generating no emissions in the matrix element region and a single emission in the parton shower region. The cross section is

$$\begin{aligned}\sigma_3^{(PS+ME)}\left(q_0\right) = \bar{\sigma}_3^{(ME)}\left(q_{\rm MS}\right)\left[\Delta_q\left(q_{\rm MS}, q_0\right)\right]^2 \Delta_g\left(q_{\rm MS}, q_0\right) \\ + \sigma_2 \times 2\left[\Delta_q\left(q_I, q_0\right)\right]^2 \int_{q_0}^{q_{\rm MS}} \mathrm{d}q\, \alpha_S(q)\, \Gamma_{q\to qg}(q)\, \Delta_g(q, q_0),\end{aligned} \tag{2.4}$$

where $\bar{\sigma}_3^{(ME)}\left(q_{\rm MS}\right)$ is the reweighted matrix element contribution for three jets resolved at the merging scale. The first term in eq. (2.4) corresponds to a single emission above the merging scale followed by parton shower evolution from the merging scale down to the hadronization scale with no resolvable emissions. The second term corresponds to no emissions above the merging scale followed by a single parton shower emission below the merging scale. In the NLL expansion of eq. (2.4), we replace $\bar{\sigma}_3^{(ME)}\left(q_{\rm MS}\right)$ by the NLL parton shower approximation in eq. (2.1). This results in a simplification of eq. (2.4)

$$\sigma_3\left(q_0\right) = \sigma_2 \times 2\left[\Delta_q\left(q_I, q_0\right)\right]^2 \int_{q_0}^{q_I} \mathrm{d}q\, \alpha_S(q)\, \Gamma_{q\to qg}(q)\, \Delta_g(q, q_0), \tag{2.5}$$

yielding the expected NLL parton shower cross section for a single resolved emission which is independent of the merging scale.

2.2 Angular-ordered CKKW merging

The merging variable used to define the jet cross sections must regulate both soft and collinear singularities, so it must be a transverse momentum measure. The merging variable in the original CKKW publication is defined in terms of the Durham jet measure [23] for two partons i and j,

$$y_{\mathrm{dur}_{ij}} = \frac{2\min\left(E_i^2, E_j^2\right)}{s}\left(1 - \cos\theta_{ij}\right), \tag{2.6}$$

where $E_{i,j}$ are the energies of the two partons, θ_{ij} is the angle between the two partons and s is the centre-of-mass-energy squared. The merging transverse momentum variable is defined by

$$k_{\perp} = \sqrt{y_{ij} s}. \tag{2.7}$$

The parton shower with which we wish to merge the matrix elements may not be ordered in transverse momentum, in which case the merging variable cannot be chosen to be the same as the evolution variable, as was assumed in section 2.1.

In the Herwig++ parton shower, splittings are described by the variables $(\tilde{q}, z, \phi)$, where $\tilde{q}$ is an angular-ordered evolution variable, z is the momentum fraction of the emitted parton and ϕ is the azimuthal angle of the branching. The evolution variable is defined by

$$z^2(1-z)^2\tilde{q}^2 = p_{\perp}^2 + (1-z)m_i^2 + zm_j^2 - z(1-z)m_{\widetilde{ij}}^2, \tag{2.8}$$

where $p_{\perp}$ is the relative transverse momentum of the branching and $m_{\widetilde{ij}}$ and $m_{i,j}$ are the masses of the parent and child partons respectively.

The probability for a branching $\widetilde{ij} \to ij$ in the phase space measure $\tilde{q} \to \tilde{q} + \mathrm{d}\tilde{q}$ is given by

$$\mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z) = \frac{\alpha_S(p_{\perp})}{2\pi} \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2} \, \mathrm{d}z \, P_{\widetilde{ij}\to ij}(z, \tilde{q}), \tag{2.9}$$

where $P_{\widetilde{ij}\to ij}(z, \tilde{q})$ is the corresponding quasi-collinear splitting function [27]. The Sudakov form factor, giving the probability of a parton shower line of flavour $\widetilde{ij}$ evolving from a scale $\tilde{q}_1$ down to $\tilde{q}_2$ undergoing no resolvable emissions, is given by

$$\Delta_{\widetilde{ij}}(\tilde{q}_1, \tilde{q}_2) = \exp\left[-\sum_{\widetilde{ij}\to ij} \int_{\tilde{q}_2}^{\tilde{q}_1} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z)\right]. \tag{2.10}$$

In order to accommodate the fact that the evolution and merging variables are not identical, the CKKW algorithm must include some additional features to that outlined in section 2.1. Changes must be made to the Sudakov form factors with which the matrix elements are reweighted and the initial conditions with which the parton shower is invoked, furthermore, when the shower is invoked, a veto must be applied to prevent it generating emissions with $k_{\perp}(\tilde{q}, z) > k_{\perp_{\mathrm{MS}}}$.

The Sudakov form factor used for the matrix element reweighting, defined in eq. (2.2), corresponds to the probability of evolving from a scale q down to the hadronization scale with no emissions resolvable at the merging scale. In the case of section 2.1, this was achieved by setting the lower limit on the integral to q_{MS}, however, now this cut, defining what is meant by a *resolvable emission*, must be implemented as a θ-function in the Sudakov form factors used in step 4. The Sudakov form factors for the reweighting are then given by

$$\Delta_{\widetilde{ij}}^{R}(\tilde{q}; k_{\perp_{\mathrm{MS}}}) = \exp\left[-\sum_{\widetilde{ij}\to ij} \int_{\tilde{q}_0}^{\tilde{q}} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}', z)\theta\left(k_{\perp}(\tilde{q}', z) - k_{\perp_{\mathrm{MS}}}\right)\right]. \tag{2.11}$$

The prescription for constructing the Sudakov weights is then identical to that in section 2.1 except for factors of z in the scale from which each child evolves, which are required for the angular-ordered evolution. Each intermediate line, connecting branchings at $(\tilde{q}_1, z_1)$ and $(\tilde{q}_2, z_2)$ in the pseudo-shower history, contributes a factor

$$\Delta^R_{\tilde{ij}}(z_1\tilde{q}_1; k_{\perp_{\rm MS}})/\Delta^R_{\tilde{ij}}(\tilde{q}_2; k_{\perp_{\rm MS}}). \tag{2.12}$$

Each external line, from a branching at $(\tilde{q}, z)$ in the pseudo-shower history, contributes a factor

$$\Delta^R_{\tilde{ij}}(z\tilde{q}; k_{\perp_{\rm MS}}). \tag{2.13}$$

2.3 Highest multiplicity treatment

The original CKKW publication did not treat the highest multiplicity matrix element contribution any differently to the other multiplicities. In refs. [9, 12] it was noted that a different treatment of highest multiplicities must be employed in order to fill the phase space in the matrix element region to all orders in α_S. Since computational limits mean that only matrix elements with up to a maximum of N partons can be calculated, the standard approach leads to a maximum of N partons being generated above the merging scale. The parton shower generates to all orders in α_S and therefore we should also let the matrix element region generate to all orders. This can be achieved by allowing the highest multiplicity channel parton shower to generate emissions in the region with $k_\perp$ less than that of the lowest transverse momentum of the matrix element emissions, $k_{\perp_L}$. This is achieved by changing the scale of the parton shower vetoes and Sudakov form factor cuts from $k_{\perp_{\rm MS}}$ to $k_{\perp_L}$.

2.4 Problems with the algorithm

The above procedure is heavily reliant on having an exact mapping between the shower variables and the merging measure $k_{\perp_{\rm MS}}$ so that the parton shower vetoes and Sudakov cuts can be correctly applied. A mapping from the momentum clustered in step 3 to the corresponding shower variables is also required, so that the correct scales for the Sudakov reweighting and initial shower conditions are obtained. In practice obtaining such mappings may be difficult due to the complexity of the shower kinematics.

The initial scale at which the parton shower is invoked is vital to the algorithm. Initiating the parton shower directly from the merging scale would result in a radiation gap, where emissions with transverse momentum less than the merging scale but evolution scale greater than the merging scale are missed. In the angular-ordered shower, this radiation corresponds to soft, wide-angle emissions. The original CKKW publication attempts to resolve this by invoking the parton shower from each external parton at a scale corresponding to the node at which it was 'created' in the pseudo-shower history. Although adopting this maximal initial scale helps fill the radiation gap, the extra soft, wide-angle radiation that results, is emitted from the external parton in the pseudo-shower history, rather than the intermediates, as implied by colour coherence [15]. The original CKKW publication argued that this should be a sub-leading effect, however, it will certainly change the colour structure of the configuration, which may cause problems when non-perturbative hadronization models are applied.

The original CKKW algorithm also assumes that the clustering of momentum in step 3 and subsequent mapping to parton shower variables results in a set of emission scales that respect the ordering of the parton shower, i.e.

$$\tilde{q}_I > \tilde{q}_1 > \ldots > \tilde{q}_n > \tilde{q}_0. \tag{2.14}$$

The clustering scheme of the original algorithm does not guarantee this.

These issues were studied in ref. [13] and found to result in problems when the parton shower was not ordered in transverse momentum. It was found that both the CKKW-L and CKKW algorithms provide a reliable merging when implemented in a parton shower in which the evolution variable is given by a transverse momentum variable that is equal to or approximates the merging variable. When the CKKW algorithm was applied to a virtuality-ordered parton shower exhibited discontinuities around the merging scale and a resultant dependence on the merging scale. These problems were most prominent in parton-level observables in the merging variable itself. While the application of the hadronization models results in some smoothing of these features, the problems were seen to persist at hadron level.

In ref. [12], a study of the algorithm with angular- and virtuality-ordered parton showers was presented. In that work, a number of *ad hoc* adaptations were applied and tuned in order to achieve a reasonably smooth merging at the parton level, nevertheless, some problems remained at the hadron level. In this article we aim to overcome these problems with a set of well motivated modifications based on the POWHEG shower reorganisation.

3 Shower reorganisation

In the POWHEG next-to-leading-order matching scheme it was shown that a general parton shower may be rearranged such that the hardest emission[6] can be generated first. The hardest emission can then be generated according to the exact NLO matrix elements such that inclusive observables are distributed according to the NLO cross section while the soft/collinear resummation of the shower is undisturbed. In the POWHEG shower reorganisation, the hardest emission is then dressed by inserting a truncated shower of wide angle, soft emissions prior to it, and a vetoed shower consisting of smaller $p_\perp$, smaller angle emissions, after it.

The CKKW algorithm generates a set of n emissions above the merging scale $y_{\rm MS}$ according to exact tree-level matrix elements up to $\mathcal{O}(\alpha_S^N)$. This defines a set of n hard emissions. In order to reproduce the full shower around this set of hard emissions we employ a generalisation of the POWHEG shower reorganisation. In the following a review of the POWHEG reorganisation is presented followed by its extension to the CKKW case. The notation used relates specifically to that of the Herwig++ shower, however the treatment is independent of the details of the parton shower.

[6]The hardest emission here refers to the emission with highest transverse momentum.

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3.1 POWHEG reorganisation

A general Herwig++ parton shower, evolving from a parton of flavour $\widetilde{ij}$ and initial scale $\tilde{q}_I$, can be represented by a generating functional $S_{\widetilde{ij}}(\tilde{q}_I)$. The evolution of the parton shower may be expressed by the recursive equation,

$$\begin{aligned}S_{\widetilde{ij}}(\tilde{q}_I) = {} & \Delta_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}_0) S_{\widetilde{ij}}(\tilde{q}_0) \\ & + \int_{\tilde{q}_0}^{\tilde{q}_I} \Delta_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}) \sum_{\widetilde{ij}\to ij} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z) S_i(z\tilde{q})\, S_j((1-z)\tilde{q}) \,. \end{aligned} \tag{3.1}$$

The first term in eq. (3.1) is a no emission term corresponding to the probability of evolving from the initial scale to hadronization scale with no resolvable emissions generated. The second term represents the probability of producing at least one emission, with the first generated at a scale $\tilde{q}$ and further showers evolving down to the hadronization scale.

It is possible to expand eq. (3.1) to explicitly show the hardest emission of the shower. The hardest emission is described by the shower variables $(\tilde{q}_h, z_h, \phi_h)$ and has an associated transverse momentum $p_{\perp_h}$. The shower may produce any number of other emissions before the hardest emission and any number of emissions after it but all of these must have $p_\perp < p_{\perp_h}$. The shower can therefore be written as

$$\begin{aligned}S_{\widetilde{ij}}(\tilde{q}_I) = {} & \Delta_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}_0) S_{\widetilde{ij}}(\tilde{q}_0) + \int_{\tilde{q}_0}^{\tilde{q}_I} \bar{S}^T_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}_h; p_{\perp_h}) \sum_{\widetilde{ij}\to ij} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z) \\ & \times \bar{S}^V_i(z_h\tilde{q}_h; p_{\perp_h})\, \bar{S}^V_j((1-z_h)\tilde{q}_h; p_{\perp_h}) \end{aligned} \tag{3.2}$$

where $\bar{S}^T$ refers to a *truncated shower* and $\bar{S}^V$ refers to a *vetoed shower*. The truncated shower in eq. (3.2) is responsible for evolving from the initial scale down to the scale of the hardest emission producing any number of emissions with transverse momentum less than $p_{\perp_h}$. The emissions within the truncated shower correspond to soft, wide angle gluon emissions, and so do not change the flavour of the shower line. The vetoed shower evolves from the scale of the hardest emission down to the hadronization scale also generating only emissions with transverse momentum less than $p_{\perp_h}$, its evolution is defined by

$$\begin{aligned}\bar{S}^V_{\widetilde{ij}}(\tilde{q}_h; p_{\perp_h}) = {} & \Delta_{\widetilde{ij}}(\tilde{q}_h, \tilde{q}_0) \bar{S}_{\widetilde{ij}}(\tilde{q}_0) + \int_{\tilde{q}_0}^{\tilde{q}_h} \Delta_{\widetilde{ij}}(\tilde{q}_h, \tilde{q}) \sum_{\widetilde{ij}\to ij} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z) \theta(p_{\perp_h} - p_\perp(\tilde{q}, z)) \\ & \times \bar{S}^V_i(z\tilde{q}; p_{\perp_h})\, \bar{S}^V_j((1-z)\tilde{q}; p_{\perp_h}) \,. \end{aligned} \tag{3.3}$$

The recursive equation describing the evolution of the truncated shower is given by

$$\begin{aligned}\bar{S}^T_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}_h; p_{\perp_h}) = {} & \Delta_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}_h) + \int_{\tilde{q}_h}^{\tilde{q}_I} \Delta_{\widetilde{ij}}(\tilde{q}_I, \tilde{q}) \mathrm{d}\mathcal{P}_{\widetilde{ij}\to\widetilde{ij}g}(\tilde{q}, z) \theta(p_{\perp_h} - p_\perp(\tilde{q}, z)) \\ & \times \bar{S}^T_{\widetilde{ij}}(z\tilde{q}, \tilde{q}_h; p_{\perp_h})\, \bar{S}^V_g((1-z)\tilde{q}; p_{\perp_h}) \,. \end{aligned} \tag{3.4}$$

The Sudakov form factors and splitting functions appearing in the eqs. (3.4) and (3.3) are identical to those in the standard shower equation of eq. (3.1) with the exception that the splitting functions in both new showers have an additional θ-function. This θ-function guarantees that no emissions with transverse momentum greater than that of the hardest emission are generated. Standard Monte Carlo techniques require that the splitting functions of a parton shower match those appearing in the Sudakov form factors. The introduction of the θ-functions mean that this is not the case for the vetoed and truncated showers in eq. (3.2), we highlight this in our notation with a bar.

In order to make the truncated and vetoed showers suitable for a Monte Carlo treatment, the original POWHEG publication [15] splits the Sudakov form factor appearing in eqs. (3.4) and (3.3) into two parts according to

$$\Delta_f(z_i\tilde{q}_i, \tilde{q}_{i+1}) = \Delta_f^V(z_i\tilde{q}_i, \tilde{q}_{i+1}; p_{\perp_h})\bar{\Delta}_f^R(z_i\tilde{q}_i, \tilde{q}_{i+1}; p_{\perp_h}). \tag{3.5}$$

Here, Δ_f^V refers to a *vetoed Sudakov* in which the exponent contains a θ-function, which matches that in the splitting function of eqs. (3.4) and (3.3) and is given by

$$\Delta_{\widetilde{ij}}^V(z_i\tilde{q}_i, \tilde{q}_{i+1}; p_{\perp_h}) = \exp\left[-\sum_{\widetilde{ij}\to ij}\int_{\tilde{q}_{i+1}}^{z_i\tilde{q}_i} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z)\theta\left(p_{\perp_h} - p_\perp(\tilde{q}, z)\right)\right]. \tag{3.6}$$

The other factor, $\bar{\Delta}_f^R$, contains the opposite θ-function and is referred to as a *remnant Sudakov* given by

$$\bar{\Delta}_{\widetilde{ij}}^R(z_i\tilde{q}_i, \tilde{q}_{i+1}; p_{\perp_h}) = \exp\left[-\sum_{\widetilde{ij}\to ij}\int_{\tilde{q}_{i+1}}^{z_i\tilde{q}_i} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z)\theta\left(p_\perp(\tilde{q}, z) - p_{\perp_h}\right)\right]. \tag{3.7}$$

The combination of the splitting functions in eqs. (3.4) and (3.3) and the vetoed Sudakov form factors result in a parton shower that may be generated with standard vetoes allowing only emissions with $p_\perp < p_{\perp_h}$, however, the presence of the remnant Sudakov form factors appears to spoil this picture. On the contrary, it turns out that the seemingly awkward remnant factors have a key role to play in formalising how to generate the hardest emission first. In ref. [15] it was proven that the hardest emission in the shower is generated along the hardest line, the line for which all $z_i > \frac{1}{2}$ and, moreover, all non-soft emissions preceding it give rise to subleading collinear contributions. This means that the truncated shower may be considered as comprising solely of soft gluon emissions and that z_i can be effectively replaced by one in all of the associated remnant factors. It is also shown that the θ-function in the remnant Sudakov form factor exponent is zero for scales less than $\tilde{q}_h$ and so the replacement $z_i \to 1$ also holds in the remnant Sudakov factors resulting from emissions after the hard emissions. The net result of these replacements is that the product of all remnant Sudakov form factors combine as a single remnant Sudakov factor:

$$\Delta_{\widetilde{ij}}^R(\tilde{q}_I, \tilde{q}_0; p_{\perp_h}) = \exp\left[-\sum_{\widetilde{ij}\to ij}\int_{\tilde{q}_0}^{\tilde{q}_I} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z)\theta\left(p_\perp(\tilde{q}, z) - p_{\perp_h}\right)\right]. \tag{3.8}$$

The POWHEG treatment which has been outlined, results in a reorganisation of the shower such that the hardest emission may be generated first. The Monte Carlo interpretation of this reorganisation is:

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1. the hardest emission (q_h, z_h, ϕ_h) is generated according to the appropriate splitting function reweighted with the remnant Sudakov factor of eq. (3.8);

2. a truncated shower, allowing only non-flavour-changing emissions with $p_\perp < p_{\perp_h}$ is initiated, evolving the shower from $\tilde{q}_I$ down to $\tilde{q}_h$;

3. the hardest emission is forced with shower variables (q_h, z_h, ϕ_h);

4. showers with a veto, allowing only emissions with $p_\perp < p_{\perp_h}$, evolve all external lines down to the hadronization scale.

A shower generated in this way should differ from the standard shower by only subleading terms.

The vetoed shower is defined by

$$S^V_{\widetilde{ij}}(\tilde{q}_h; p_{\perp_h}) = \Delta^V_{\widetilde{ij}}(\tilde{q}_h, \tilde{q}_0) S^V_{\widetilde{ij}}(\tilde{q}_0) + \int_{\tilde{q}_0}^{\tilde{q}_h} \Delta^V_{\widetilde{ij}}(\tilde{q}_h, \tilde{q}) \sum_{\widetilde{ij}\to ij} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q}, z)\theta(p_{\perp_h} - p_\perp(\tilde{q}, z)) \times S^V_i(z\tilde{q}; p_{\perp_h})\, S^V_j((1-z)\tilde{q}; p_{\perp_h}) \,. \tag{3.9}$$

and corresponds to a standard shower with vetoes applied such that only emissions with $p_\perp < p_{\perp_h}$ are generated. The truncated shower is defined by

$$S^T_i(\tilde{q}_I, \tilde{q}_h; p_{\perp_h}) = \Delta^V_i(\tilde{q}_I, \tilde{q}_h) + \int_{\tilde{q}_h}^{\tilde{q}_I} \Delta^V_i(\tilde{q}_I, \tilde{q}) \mathrm{d}\mathcal{P}_{i\to ig}(\tilde{q}, z)\theta(p_{\perp_h} - p_\perp(\tilde{q}, z)) \times S^T_i(z\tilde{q}, \tilde{q}_h; p_{\perp_h})\, S^V_g((1-z)\tilde{q}; p_{\perp_h}) \,. \tag{3.10}$$

and corresponds to a standard vetoed parton shower line, constrained not to produce any flavour changing emissions, that is stopped once the truncated line has evolved down to the scale $\tilde{q}_h$.

3.2 CKKW shower reorganisation

In the POWHEG treatment reviewed in section 3.1 a single hardest emission is separated such that it may be corrected with matrix elements. In the CKKW algorithm we aim to improve the parton shower with tree-level matrix elements for all parton multiplicities resolved at the merging scale $k_{\perp_{\rm MS}}$. We perform a reorganisation of the parton shower, analogous to the POWHEG reorganisation, splitting the shower into two parts: a *hard shower* describing emissions resolved above the merging scale; and another shower producing the rest of the shower emissions around this hard shower. The hard shower can then be generated according to the tree-level matrix elements as required by the CKKW algorithm.

The result of this generalisation of the POWHEG reconstruction is a set of truncated and vetoed showers which fill in the radiation between the hard emissions defined by the hard shower history. In order to see how this works we first consider the next step up from the POWHEG case of a single hard emission, where we have exactly two hard emissions along the hard shower line, generated at scales $\tilde{q}_1$ and $\tilde{q}_2$. One possible configuration of this hard shower line is given in figure 1. As was done in formulating the POWHEG scheme,

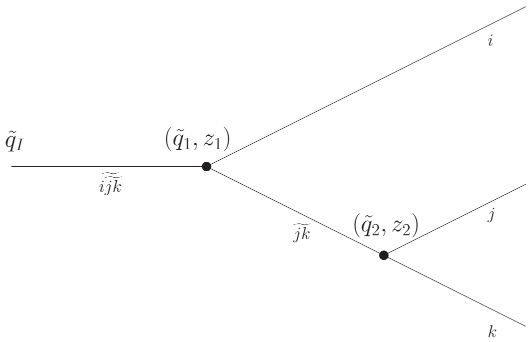

Figure 1. An example of a hard shower line configuration where two emissions are generated above $k_{\perp\mathrm{MS}}$.

the full parton shower can be constructed around this hard shower line by constructing an equation analogous to eq. (3.2),

$$
\begin{aligned}
S^{(2)}_{\widetilde{\widetilde{ij}k}}(\tilde{q}_I) = \int_{\tilde{q}_0}^{\tilde{q}_I} & \bar{S}^T_{\widetilde{\widetilde{ij}k}}(\tilde{q}_I, \tilde{q}_1; k_{\perp\mathrm{MS}})\, \mathrm{d}\mathcal{P}_{\widetilde{\widetilde{ij}k} \to i\widetilde{jk}}(\tilde{q}_1, z_1) \bar{S}^V_i\left((1-z_1)\tilde{q}_1; k_{\perp\mathrm{MS}}\right) \qquad (3.11)\\
& \times \int_{\tilde{q}_0}^{\tilde{q}_1} \bar{S}^T_{\widetilde{jk}}(z_1\tilde{q}_1, \tilde{q}_2; k_{\perp\mathrm{MS}})\, \mathrm{d}\mathcal{P}_{\widetilde{jk}\to jk}(\tilde{q}_2, z_2) \bar{S}^V_j\left(z_2\tilde{q}_2; k_{\perp\mathrm{MS}}\right) \bar{S}^V_k\left((1-z_2)\tilde{q}_2; k_{\perp\mathrm{MS}}\right).
\end{aligned}
$$

The superscript (2) on S denotes that this does not describe a general shower line, but the subset of shower lines with exactly two emissions above the merging scale. Eq. (3.11) contains two truncated showers, one containing parton shower emissions with $k_\perp < k_{\perp\mathrm{MS}}$ before the hard emission $(\tilde{q}_1, z_1)$ and the other containing emissions with $k_\perp < k_{\perp\mathrm{MS}}$ between the hard emissions at $(\tilde{q}_1, z_1)$ and $(\tilde{q}_2, z_2)$. The truncated and vetoed showers are given by eq. (3.4) and eq. (3.3),[7] respectively.

As in the POWHEG case, the splitting functions and Sudakov form factors for the truncated and vetoed showers in eq. (3.11) do not match each other and are therefore not suitable for a standard Monte Carlo treatment. However, we can use the same manipulations as in the POWHEG formulation to split the Sudakov form factors into a product of a Sudakov form factor that matches the vetoed splitting functions and a remnant Sudakov form factor, as in eq. (3.5). The truncated showers contain only soft radiation so again we may set $z_i \to 1$ in the remnant Sudakov form factor of eq. (3.7) with only subleading differences. The result of this is that the product of remnant Sudakov form factors for a particular truncated or vetoed line combine to give a remnant Sudakov factor. Rather than resulting in a single remnant Sudakov factor as in the POWHEG scheme, we now get a product of remnant Sudakov factors. The product of remnant Sudakov factors for the hard shower configuration of figure 1 is given by

$$
\frac{\Delta^R_{\widetilde{\widetilde{ij}k}}(\tilde{q}_I; k_{\perp\mathrm{MS}})}{\Delta^R_{\widetilde{\widetilde{ij}k}}(\tilde{q}_1; k_{\perp\mathrm{MS}})} \Delta^R_i((1-z_1)\tilde{q}_1; k_{\perp\mathrm{MS}}) \frac{\Delta^R_{\widetilde{jk}}(z_1\tilde{q}_1; k_{\perp\mathrm{MS}})}{\Delta^R_{\widetilde{jk}}(\tilde{q}_2; k_{\perp\mathrm{MS}})} \Delta^R_j(z_2\tilde{q}_2; k_{\perp\mathrm{MS}}) \Delta^R_k((1-z_2)\tilde{q}_2; k_{\perp\mathrm{MS}}),
$$

[7]With the replacement $p_\perp \to k_\perp$. The variable $p_\perp$ is the shower definition of transverse momentum and $k_\perp$ is the merging variable transverse momentum measure.

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where the remnant Sudakov factor is given by

$$\Delta^R_{\widetilde{ij}}(\tilde{q}; k_{\perp_{\rm MS}}) = \exp\left[-\sum_{\widetilde{ij}\to ij}\int_{\tilde{q}_0}^{\tilde{q}} \mathrm{d}\mathcal{P}_{\widetilde{ij}\to ij}(\tilde{q},z)\theta\left(k_\perp(\tilde{q},z) - k_{\perp_{\rm MS}}\right)\right]. \tag{3.12}$$

In the CKKW algorithm the hard shower is generated by choosing a jet multiplicity n as described in section 2.1 and generating n parton momenta according to the appropriate jet cross section. A pseudo-shower history and corresponding shower variables are then assigned by applying a clustering algorithm to the n parton momenta, until they are clustered back to a leading order configuration. The shower reorganisation presented here results in a product of remnant Sudakov factors, with which these hard shower configurations should be reweighted. These remnant Sudakov factors can generally be found from the pseudo-shower history by applying the following prescription:

- each internal line from a branching at (q_1, z_1) to (q_2, z_2) contributes a factor

$$\frac{\Delta^R_f(z_1\tilde{q}_1; k_{\perp_{\rm MS}})}{\Delta^R_f(\tilde{q}_2; k_{\perp_{\rm MS}})}; \tag{3.13}$$

- each external line from a branching at (q, z) contributes a factor

$$\Delta^R_f(z\tilde{q}; k_{\perp_{\rm MS}}). \tag{3.14}$$

These remnant Sudakov factors match the Sudakov factors in eq. (2.11) that we argued should be introduced in order to extend the CKKW procedure for transverse momentum showers to the angular-ordered shower.

The parton shower for emissions below the cut is generated by producing truncated and vetoed showers around the hard shower according to the following prescription:

- each internal line from a branching at (q_1, z_1) to (q_2, z_2) results in a truncated shower

$$S^T_f\left(z_1\tilde{q}_1, \tilde{q}_2; k_{\perp_{\rm MS}}\right); \tag{3.15}$$

- each external line from a branching at (q, z) results in a vetoed shower

$$S^V_f\left(z\tilde{q}; k_{\perp_{\rm MS}}\right). \tag{3.16}$$

The formalism presented in ref. [20] yields the same result that we have arrived at, where showers describing emissions in two disjoint phase space regions are combined using truncated and vetoed showers. In the case of final-state parton showers this result comes directly from splitting the evolution kernels into two terms, describing the emission probability into the two regions. The two kernels each contain a corresponding θ-function leading to the equivalent of eq. (3.5). It is assumed that the arguments of the Sudakov form factors depend only on the evolution variables and not on the auxiliary splitting variables. This leads directly to the resultant shower reorganisation by consideration of the no emission probabilities.

Angular-ordered parton shower evolution dictates that the auxiliary splitting variable z, enter the arguments of the Sudakov form factors as defined in eq. (3.1). This complication necessitates the more thorough treatment of the parton shower decomposition, which we have presented, to show that the same reorganisation results in only subleading differences in the parton shower resummation. The formalism presented here also makes clear the connection to the POWHEG method, in which truncated showers were first discussed.

4 The algorithm

In order to implement the procedure described in section 3.2 we employ the strategy of ref. [18], where the hardest emissions, or set of hard emissions in this case, are interpreted as parton shower emissions. This approach leads to a straightforward implementation of the truncated showers, where a truncated shower, evolving between hard emissions at $(\tilde{q}_1, z_1, \phi_1)$ and $(\tilde{q}_2, z_2, \phi_2)$, is generated by initiating a standard parton shower at $z_1\tilde{q}_1$ with vetoes allowing only non-flavour-changing emissions with $k_\perp < k_{\perp_{\rm MS}}$ and stopping the truncated shower once it has evolved beyond $\tilde{q}_2$, at which point the second hard emission is forced with splitting variables $(\tilde{q}_2, z_2, \phi_2)$. This allows the full shower of truncated showers, hard emissions and vetoed showers to be generated as a single shower evolution from the leading-order configuration. This results in a substantial improvement over earlier CKKW implementations with angular-ordered showers [7, 12], since now the *colour structure* in the event is plainly equivalent to that which the shower would have produced by default i.e. it respects colour coherence.

In order to interpret the matrix element emissions as shower emissions, we require an exact mapping from the set of n external parton momentum and assigned pseudo-shower history to a set of shower splitting variables, $(\tilde{q}, z, \phi)$, describing each emission. Obtaining such a mapping equates to inverting the momentum reconstruction, which is performed at the end of the standard parton shower to translate the set of shower variables into the parton momenta, and is described in section 4.2. Having such a mapping also provides the exact shower variables that are to used for the Sudakov and α_S reweighting.

The full modified CKKW algorithm is described below.

1. The jet multiplicity n is generated with probability

$$P_n = \frac{\sigma_n(k_{\perp_{\rm MS}})}{\sum_N \sigma_i(k_{\perp_{\rm MS}})}, \tag{4.1}$$

   where cross sections are evaluated at a fixed strong coupling $\alpha_{S_{\rm ME}}$.

2. The n external parton momenta are generated according to $\mathrm{d}\sigma(k_{\perp_{\rm MS}})$.

3. Pairs of external parton momenta are clustered[8] down to a leading-order configuration, assigning a pseudo-shower history.

[8]The clustering procedure is discussed in section 4.4.

4. The inverse momentum reconstruction is applied to the external momenta and shower history such that a set of shower splitting variables $(\tilde{q}, z, \phi)$ are found, describing $n-2$ hard branchings.

5. The configuration is reweighted to include the Sudakov form factors and running α_S. This corresponds to assigning the configuration a weight W and rejecting the configuration if $W < \mathcal{R}$.[9] The weight is constructed from the pseudo-shower history, according to the following prescription:

   - each hard emission at $(\tilde{q}, z)$ contributes a running α_S factor

$$\frac{\alpha_S\left(p_\perp(\tilde{q}, z)\right)}{\alpha_{S_{\mathrm{ME}}}}; \tag{4.2}$$

   - each internal line between hard emissions at $(\tilde{q}_1, z_1)$ to $(\tilde{q}_2, z_2)$ contributes a Sudakov factor

$$\frac{\Delta_f^R(z_1\tilde{q}_1; k_{\perp_{\mathrm{MS}}})}{\Delta_f^R(\tilde{q}_2; k_{\perp_{\mathrm{MS}}})}; \tag{4.3}$$

   - each external line from a hard emission at $(\tilde{q}, z)$ contributes a Sudakov factor

$$\Delta_f^R(z\tilde{q}; k_{\perp_{\mathrm{MS}}}). \tag{4.4}$$

   If the configuration is rejected[10] return to step 1.

6. Parton shower lines are initiated from the leading-order configuration which are to be evolved according to the procedure:

   (a) If a hard emission exists at a lower scale on the shower line, then the shower is evolved as a truncated shower otherwise proceed with step (6c). The truncated showers evolve as the standard parton shower with vetoes allowing only non-flavour-changing-emissions with $k_\perp < k_{\perp_{\mathrm{MS}}}$. Each truncated emission generates a soft gluon which should be evolved according to step (6c).

   (b) Once the scale of the next hard emission is reached, the hard emission is forced creating two further shower lines, each of which should be evolved according to step (6a).

   (c) Vetoed showers evolve all external shower lines down to the hadronization scale, with vetoes allowing only emissions with $k_\perp < k_{\perp_{\mathrm{MS}}}$.

[9] $\mathcal{R}$ refers to a random number, generated in the interval $[0, 1]$.

[10] This reweighting procedure relies on the weight generated in this step satisfying $W < 1$. The fixed strong coupling used in the matrix elements α_{ME} can be chosen to be large enough that the α_S weight is always less than one. Individual Sudakov form factors are also guaranteed to be less than one while the ratio of Sudakov form factors contributed by intermediate lines must be less than one due to the angular ordering condition $z_i\tilde{q}_i > \tilde{q}_{i+1}$.

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The above scheme is adapted for the highest multiplicity channel, where $n = N$, by the replacement $k_{\perp_{\rm MS}} \to k_{\perp_l}$ in the shower vetoes and Sudakov form factors.

The merging algorithm presented is constructed such that the NLL resummation of the parton shower is undisturbed and therefore the dependence on the merging scale cancels at the NLL level. That this is true, follows directly from the shower reorganisation presented in section 3 and the fact that the tree-level cross sections, describing emissions in the matrix element region, may be approximated at NLL accuracy by the product of the leading order cross section and the corresponding set of parton shower splitting functions. All observable quantities are therefore independent of the merging scale to NLL order. This cancellation of merging scale dependence is illustrated in appendix A for the three-jet emission rate.

4.1 Shower kinematics

In order to implement the inverse momentum reconstruction and parton shower vetoes, the kinematics of the Herwig++ final-state parton shower must be understood. The full details of this are given in ref. [2] and we present a review here for completeness and to introduce our notation.

Each external parton from the hard sub-process with momentum p_J is interpreted as a progenitor for a parton shower jet. Each parton shower jet evolves from the initial scale $\tilde{q}_I = \sqrt{s}$ down to the hadronization scale $\tilde{q}_0$ undergoing a series of branchings, each described by the shower variables $(\tilde{q}, z, \phi)$.

Once all of the shower lines have evolved down to the hadronization scale, the shower evolution is stopped and the momentum of all external and intermediate partons are reconstructed from the shower variables. This is done in the centre-of-mass frame via the Sudakov decomposition, where the momentum of a parton in the shower is written

$$q_i = \alpha_i p_J + \beta_i n_J + q_{\perp i}, \tag{4.5}$$

where the reference vector n_J is taken to be a light-like vector with three-momentum equal to that of the colour partner to the jet progenitor and $q_\perp$ is the component of momentum transverse to both p_J and n_J. The momentum fraction z of each branching is defined by

$$z = \frac{\alpha_i}{\alpha_{\widetilde{ij}}}, \tag{4.6}$$

the scale of the emission $\tilde{q}$ is defined in eq. (2.8). The relative transverse momentum of the branching is defined by

$$p_{\perp i} = q_{\perp i} - z q_{\perp \widetilde{ij}}. \tag{4.7}$$

The $p_{\perp i}$ vector is written in terms of the azimuthal angle ϕ

$$p_\perp = (|p_\perp| \cos\phi, |p_\perp| \sin\phi, 0; 0)\,. \tag{4.8}$$

The Sudakov variables α_i, β_i and $q_{\perp i}$ are calculated from the shower variables recursively for all partons in the shower jet and the momentum are constructed according to eq. (4.5).

The momentum reconstruction procedure results in the progenitor momenta q_J gaining some off-shell momentum and leads to the loss of global momentum conservation. Longitudinal boosts are applied to each shower jet to restore momentum conservation while

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disturbing the internal structure of each jet as little as possible. These reshuffling boosts are defined for each jet by the transformation

$$\left(\vec{q}_J; \sqrt{\vec{q}_J^{\,2} + q_J^2}\right) \to \left(k\vec{p}_J; \sqrt{k^2\vec{p}_J^{\,2} + q_J^2}\right) \equiv q_J'. \tag{4.9}$$

Momentum conservation is ensured by requiring that the rescaling parameter k satisfies

$$\sum_J \sqrt{k^2\vec{p}_J^{\,2} + q_J^2} = \sqrt{s}. \tag{4.10}$$

4.2 Inverse momentum reconstruction

In the CKKW algorithm a clustering procedure is applied to a configuration of external parton momenta, defining a pseudo-shower history for producing that configuration as a set of branchings from a leading-order parton configuration, which in this case would be $q\bar{q}$. We aim to interpret this branching history as a set of shower branchings such that when this set of branchings are forced, the momentum reconstruction will reconstruct the original parton momenta. In order to map the shower history to a set of shower variables the momentum reconstruction procedure must be inverted; this requires two steps. First, the boost applied to each shower jet in order to conserve global momentum must be found and its inverse applied to the momenta of the shower jet. Second, the resulting momenta are decomposed into the shower variables according to eq. (4.5).

The momenta of the set of progenitors defined by the branching q_J' correspond to the momenta on the right hand side of eq. (4.9). The original on-shell progenitors p_J are related to q_J' by

$$p_J = \left(\frac{\vec{q}_J^{\,\prime}}{k}; \sqrt{\frac{\vec{q}_J^{\,\prime 2}}{k^2} + m_J^2}\right), \tag{4.11}$$

where m_J is the on-shell mass of the jet progenitor. The set of on-shell progenitors respect global momentum conservation therefore we can find the boost parameter k by solving

$$\sum_J \sqrt{\frac{\vec{q}_J^{\,\prime 2}}{k^2} + m_J^2} = \sqrt{s}. \tag{4.12}$$

Once k is found, the reference vector p_J is given by eq. (4.11); similarly n_J is given by

$$n_J = \left(\frac{\vec{q}_{\bar{J}}^{\,\prime}}{k}; \sqrt{\frac{\vec{q}_{\bar{J}}^{\,\prime 2}}{k^2}}\right), \tag{4.13}$$

where $\bar{J}$ refers to the colour partner jet of the shower jet J. In the reconstruction procedure, the Sudakov parameters of the progenitor partons are set to $\alpha_0 = 1$ and $q_{\perp 0} = 0$. Furthermore, eq. (4.5) implies that

$$\beta_0 = \frac{q_J^2 - m_J^2}{2p \cdot n}. \tag{4.14}$$

Since q'_J and q_J are related by a boost, we also have $q_J^2 = q_J'^2$. The momentum of the reconstructed progenitors q_J can then be constructed according to eq. (4.5). This defines the reshuffling boost as in eq. (4.9). The boosts for all shower jets can then be calculated, inverted and applied to all momenta in each jet. The momentum can then be decomposed into Sudakov parameters and the shower variables $(\tilde{q}, z, \phi)$ for each branching calculated from eqs. (4.6)–(4.8).

4.3 Shower vetoes

The vetoes that are applied to the truncated and vetoed showers and the cuts applied to the remnant Sudakov form factors require a mapping between the shower variables, $(\tilde{q}, z)$ and the merging scale transverse momentum measure $k_\perp$. The merging variable, for an emission $\widetilde{ij} \to ij$, is defined in some jet measure according to, $k_\perp = \sqrt{y_{ij} s}$. We have implemented the merging algorithm with the Durham [23] and LUCLUS [24] jet measures, defined by

$$y_{\mathrm{dur}_{ij}} = \frac{2\mathrm{min}\left(E_i^2, E_j^2\right)}{s} \left(1 - \cos\theta_{ij}\right), \tag{4.15}$$

$$y_{\mathrm{luc}_{ij}} = \frac{2\left(E_i E_j\right)^2}{s\left(E_i + E_j\right)^2} \left(1 - \cos\theta_{ij}\right). \tag{4.16}$$

In order to implement these vetoes a mapping between the shower variables $(\tilde{q}, z, \phi)$ and y_{ij}, in the chosen jet measure, must be found. The Herwig++ shower produces off-shell intermediate states and therefore a set of boosts must be applied to each shower line in order to ensure momentum conservation. Since the boosts depend on the full shower history, an exact mapping between the shower variables and the merging variable cannot be found. We use a mapping that is exact for a single shower emission and should give a good approximation for larger numbers of emissions. For clarity in the following, we treat partons as massless while in our implementation parton masses are retained.

In the case of $e^+e^- \to$ hadrons we have two shower progenitor partons $q(p_a)$ and $\bar{q}(p_b)$. The momenta of these progenitors, in the centre-of-mass frame, can be written,

$$p_{a,b} = \frac{\sqrt{s}}{2}\left(0, 0, \pm 1; 1\right). \tag{4.17}$$

We now consider a single gluon emission along the quark line such that $q(p_a) \to q(q_1)g(q_2)$. This is the first emission so we have $\alpha_1 = z$ and $\alpha_2 = 1 - z$. The transverse momentum of the emitted partons are given by $q_{\perp 1,2} = \pm p_\perp$. The emitted partons are considered to be external partons and therefore their momenta should be set on-shell. The β parameters in eq. (4.5) are therefore given by

$$\beta_1 = \frac{p_\perp^2}{zs}, \tag{4.18}$$

$$\beta_2 = \frac{p_\perp^2}{(1-z)s}. \tag{4.19}$$

The vetoes should correspond to vetoes on the reshuffled momenta that have had the boosts, defined in eq. (4.9), applied to them. We should therefore solve eq. (4.10) and calculate these boosts before applying eqs. (4.15) and (4.16). The reconstructed progenitor momenta are given by $q_a = q_1 + q_2$ for the quark jet and $q_b = p_b$ for the anti-quark jet. Inserting these momenta into eq. (4.10) yields the solution

$$k = 1 - \frac{p_\perp^2}{sz(1-z)}, \tag{4.20}$$

for the boost parameter. The reshuffling boost for the quark line is then defined by eq. (4.9). It follows that the three-vector of the shuffled quark progenitor ${q_a}'$ should be given by

$$\vec{q_a}' = \frac{\sqrt{s}}{2}\left(1 - \frac{p_\perp^2}{sz(1-z)}\right)(0,0,1)\,. \tag{4.21}$$

The expression in eq. (4.21) is identical to $\vec{q}_a$, as defined by $\alpha_{1,2}$ and $\beta_{1,2}$, and therefore the boost to be applied to the quark jet is the unit matrix. The shuffled momenta for the emitted partons have now been constructed and we can apply eqs. (4.15) and (4.16) to give expressions for the jet measures used to define the merging scale. These give

$$y_{\rm dur} = \min\left[z + \frac{p_\perp^2}{sz}, (1-z) + \frac{p_\perp^2}{s(1-z)}\right]^2 \frac{(1-\cos\theta)}{2}, \tag{4.22}$$

$$y_{\rm luc} = \left[\frac{\left(p_\perp^2 + (1-z)^2 s\right)\left(p_\perp^2 + z^2 s\right)}{p_\perp^2 s + s^2 z(1-z)}\right]^2 \frac{(1-\cos\theta)}{2}, \tag{4.23}$$

where

$$\cos\theta = 1 - \frac{2p_\perp^2 s}{\left(p_\perp^2 + (1-z)^2 s\right)\left(p_\perp^2 + z^2 s\right)}. \tag{4.24}$$

These mappings allow a transverse momentum measure, $k_\perp = \sqrt{ys}$, to be calculated in the merging variable for each parton shower emission. Parton shower vetoes and Sudakov cuts can then be applied by comparing this measure to the merging scale.

4.4 Clustering scheme

The parton shower decomposition presented in section 3.2 relied on our ability to interpret the series of hard branchings, defined by the matrix element momenta and assigned pseudo-shower history, as a parton shower. The inverse momentum reconstruction procedure ensures that, given an assigned pseudo-shower history, a set of parton shower emissions are found that will exactly reproduce the matrix element momenta.

Section 3.2 assumes that the assigned history is angular-ordered; we therefore aim to assign histories that obey the angular-ordering condition

$$\tilde{q}_i z_i > \tilde{q}_{i+1}, \tag{4.25}$$

for all emissions along all shower lines.

The inverse momentum reconstruction allows us to find the shower variables of all branchings in a particular pseudo-shower history. We can therefore determine whether a history is angular-ordered by following all shower lines outwards from the hard process and explicitly checking that all of the branchings satisfy eq. (4.25).

We employ a clustering procedure that creates all possible pseudo-shower histories and attempts to choose the angular-ordered history that the parton shower was most likely to produce.

It may appear that the best way to choose a history is in a probabilistic way according to the associated shower probability, as formed from the Sudakov form factor and α_S weights.[11] Attempting such a procedure in an angular-ordered shower however would involve applying the shower kinematics to regions in which they are not valid. This can result in finite probabilities being assigned to histories that the shower would never produce. Furthermore, employing a probabilistic choice results in a finite probability of unnatural shower histories being assigned, which can result in technical problems in applying hadronization models.

We therefore adopt a winner-takes-all strategy, accepting the pseudo-shower history with the smallest scalar sum of the transverse momentum of the emissions. This ensures that the history containing a set of emissions that are closest to the enhanced soft and collinear regions of phase space is chosen. In appendix B, we present some checks of this approach for further justification of this procedure.

Events for which there is no angular-ordered shower interpretation correspond to configurations that occur in a region of phase space that is inaccessible to the parton shower. In the matrix element region we aim to improve the parton shower description by covering the full phase space of emissions and therefore such configurations should be retained. We therefore choose to force the shower to generate non-angular-ordered emissions in the case that no angular-ordered histories are found. Since such non-angular-ordered emissions are manifestly subleading, the inclusion of such events does not affect the NLL resummation of the parton shower.

In practice the contribution of such events is small (generally $< 1\%$) and all the observable quantities tested were found to be insensitive to their treatment (whether such events were retained or discarded).

The full clustering procedure is:

1. all possible shower histories are created by clustering all pairs of partons whose flavours correspond to allowed branchings;

2. non-angular-ordered histories are discarded;

3. the angular-ordered history for which the scalar sum of the transverse momentum of its branchings,
$$\sum_{\text{hard emissions}} |p_\perp(\tilde{q}, z)| \,, \tag{4.26}$$
is smallest is chosen;

4. if no angular-ordered histories are found, the unordered history for which eq. (4.26) is smallest is chosen.

[11]This is the procedure used to assign pseudo-shower-histories in the CKKW-L algorithm in the colour dipole model.

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5 Results

In this section we present the results of the implementation of the modified CKKW algorithm for the process $e^+e^- \to$ hadrons at a centre-of-mass energy of 91.2 GeV at both parton and hadron level. The parton level results provide a test of the algorithm's ability to provide a smooth merging between the matrix element and parton shower regions of phase space, showing features that may be hidden by the addition of a hadronization model. The hadron level results provide the ultimate test of the algorithm's ability to describe data and in particular are sensitive to the parton colour structure assignment which we expect the modified algorithm to improve with respect to traditional CKKW methods.

A key test of the merging algorithm is its insensitivity to changes in the merging scale and merging variable. The algorithm was implemented with two merging variables: the Durham and LUCLUS jet resolution variables. For each merging variable, merging scales of $y_{\rm MS} = 5 \times 10^{-2}$, $y_{\rm MS} = 10^{-2}$ and $y_{\rm MS} = 5 \times 10^{-3}$ were used. Samples of events with all partons separated by $y > y_{\rm MS}$ were generated using MadGraph/MadEvent[25, 26] for the process with up to five partons in the final state.[12]

5.1 Parton level results

We present the distributions of the merging variable itself since these should be the most sensitive to problems with the merging procedure. In order to provide a direct comparison to ref. [13], we first present a systematic look at the algorithm with the maximum multiplicity set to three, so that the matrix element region is responsible for, at most, a single hard emission.

Figure 2 shows distributions of the scale at which three jets are resolved in the Durham jet measure for the three chosen merging scales with the Durham jet measure as the merging variable. Jet analyses were performed with the KtJet package [28]. Each of the merging scale choices exhibit a smooth transition between the two phase space regions, there also appears to be little dependence on the choice of merging scale. The CKKW distributions all closely match the matrix element correction distributions. This is to be expected since both algorithms aim to improve the parton shower distributions by applying a correction based on the three-jet matrix elements. The slight differences seen may be attributed to differences in the way the two algorithms apply this correction.

Figure 3 shows the same distributions as figure 2 but with the truncated shower switched off. Switching off the truncated shower results in a radiation gap, meaning that emissions that would be generated at scales greater than that of the hard emission, but with transverse momentum less than that of the hard emission, are never produced. This radiation gap corresponds to a deficit in the amount of soft wide-angle emissions produced from the three-jet samples. Additional parton shower emissions (on top of the hardest emission) have the effect of smearing the y_{23} distribution, as can be seen in figure 2, where the two-jet contribution is significantly displaced into the three-jet region. Comparing figures 2 and 3, we see that without the extra soft radiation produced in the truncated shower, the smearing of the two-jet region is not compensated for in the three-jet region

[12] The maximum multiplicity for each merging scale was decided according to the phase space available in the matrix element region.

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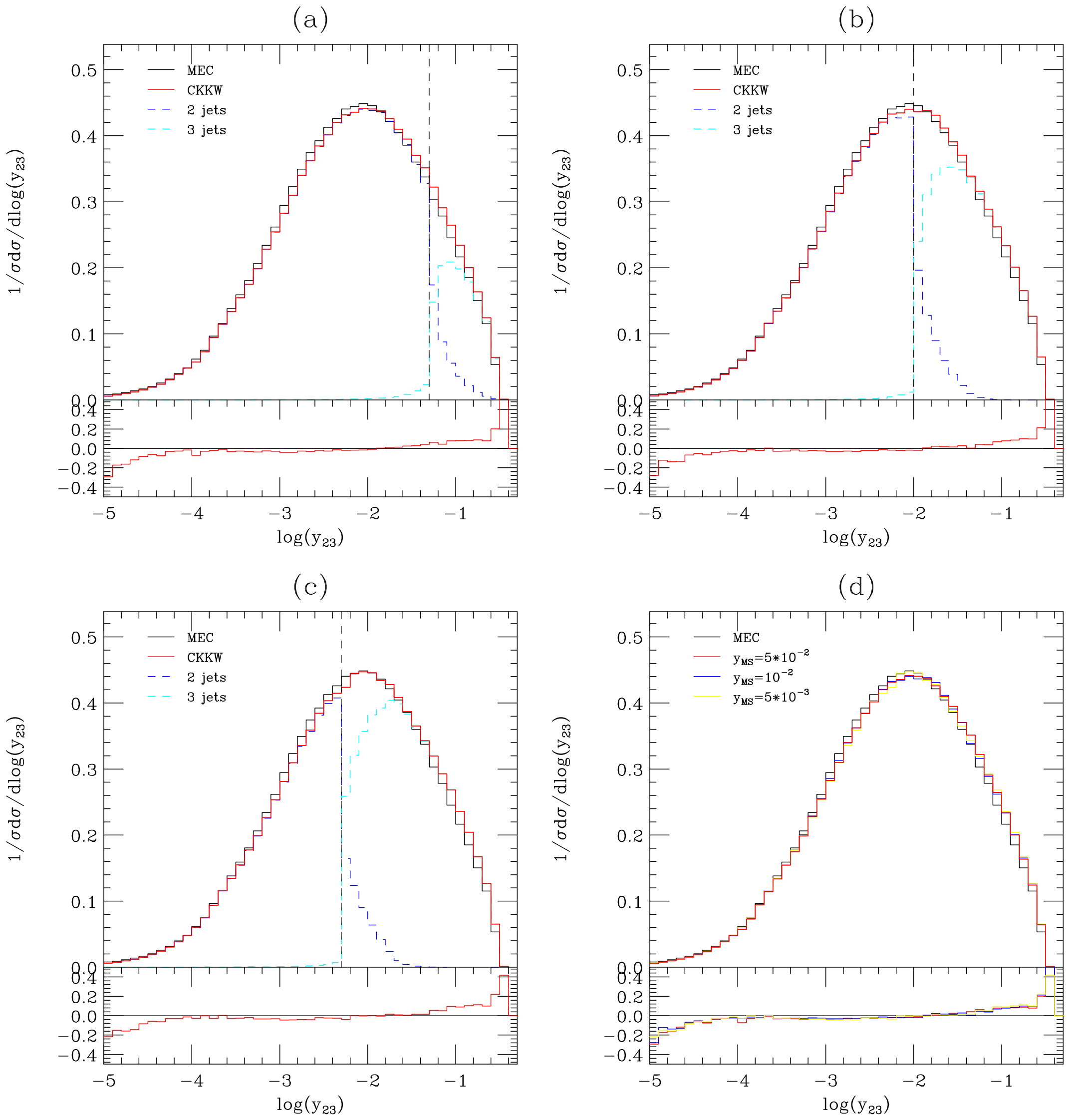

Figure 2. Parton level distributions of the scale at which three jets are resolved in the Durham jet measure for $e^+e^- \to$ hadrons at $\sqrt{s} = 91.2\,\mathrm{GeV}$. The red line shows the CKKW distribution with maximum multiplicity set to three and the black line shows the Herwig++ parton shower distribution with a matrix element correction. The blue and cyan lines show the two- and three-jet contributions to the CKKW distribution. Plots (a)-(c) show the CKKW distributions with merging scales set to $y_{\rm MS} = 5 \times 10^{-2}$, $y_{\rm MS} = 10^{-2}$ and $y_{\rm MS} = 5 \times 10^{-3}$ in the Durham jet measure. Plot (d) shows a comparison of the CKKW distributions at the different merging scale choices. The lower panel in all plots shows the difference between the CKKW and matrix element correction lines, (CKKW − MEC)/MEC.

and we observe a surplus in the three-jet region close to the merging scale. This effect is more pronounced as the merging scale is lowered and the truncated shower becomes more important. This problem will become more serious as higher multiplicity contributions are included and underlines the importance of the truncated shower in the merging algorithm.

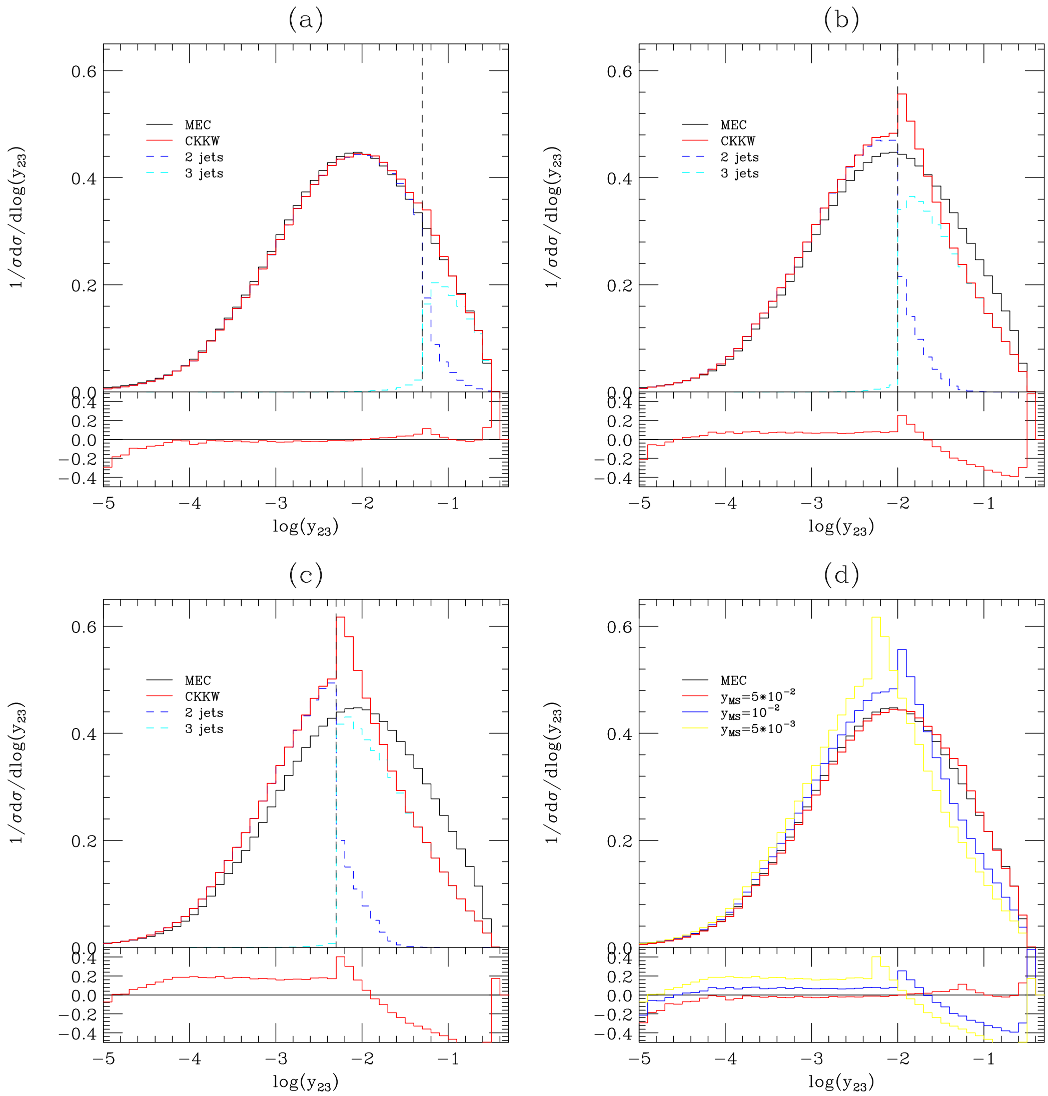

Figure 3. The same distributions as in figure 2 but with the truncated shower switched off in the CKKW treatment.

Figure 4 shows the same distributions as figure 2 but with the highest multiplicity treatment switched off. The result of switching off the highest multiplicity treatment is that a maximum of three emissions may be generated in the matrix element region. This violates the all-orders-in-α_S resummation of the parton shower. The effect of this is twofold: first, there is a deficit in the radiation generated in the three-jet channel; second, the three-jet channel receives too great a Sudakov suppression. The main observable effect in figure 4 is a surplus in the three-jet region of the distribution close to the merging scale. This can again be attributed to the deficit in radiation in the three-jet region, preventing the smearing seen in the two-jet region being compensated by that in the three-jet region. The suppression of the three-jet region also results in the relative contribution of the two-jet region being too large and we see distributions that are peaked around the merging scale and have a large dependence on the choice of merging scale.

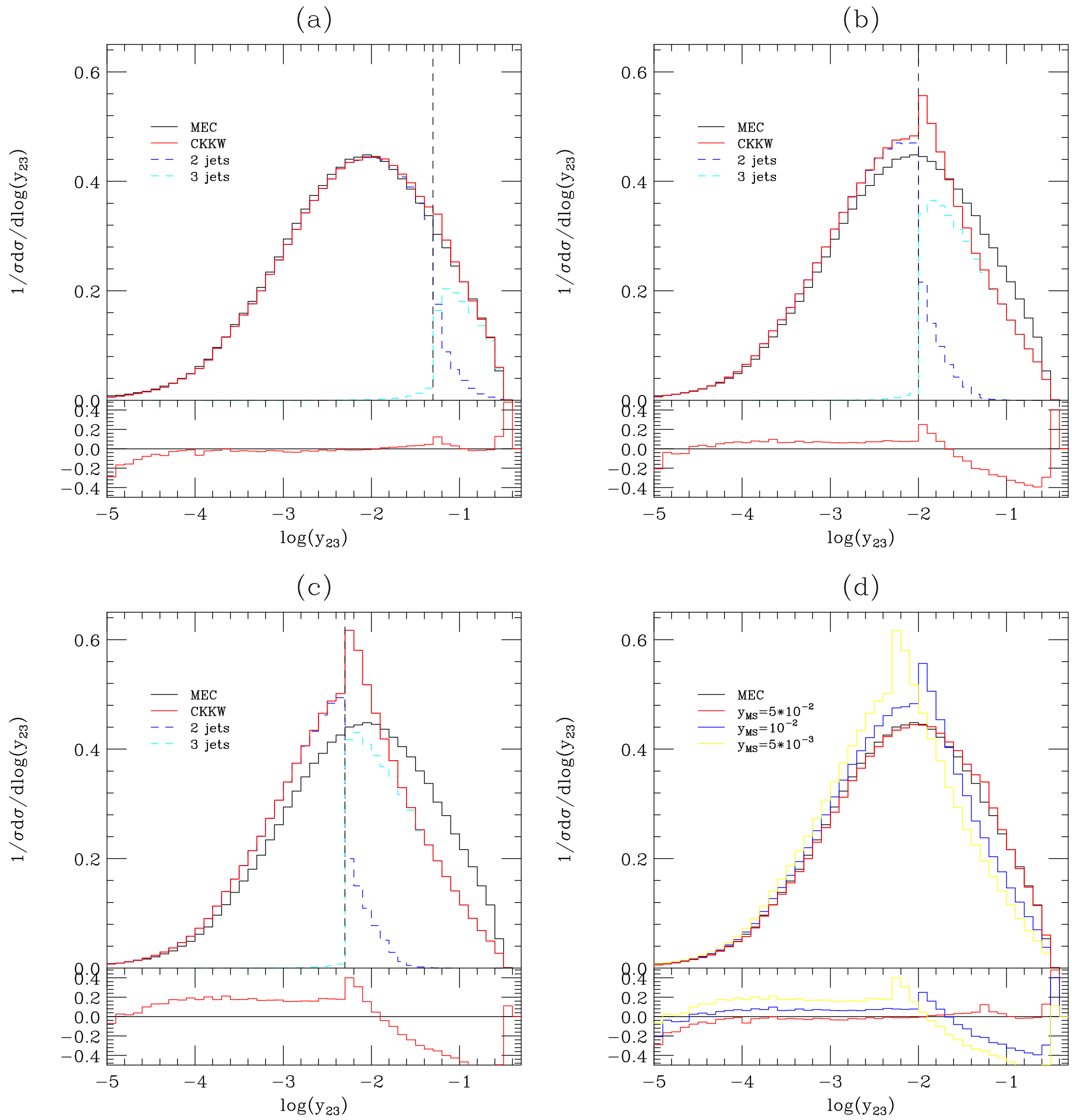

Figure 4. The same distributions as in figure 2 but with the highest multiplicity treatment switched off in the CKKW treatment.

Figures 5 and 6 show the distributions of the scale at which three jets are resolved, for the algorithm with maximum multiplicity set to up to five jets, with the merging algorithm defined in the Durham and LUCLUS jet measures respectively. As in figure 2, all distributions appear to be smooth around the merging scale and to be relatively insensitive to the choice of merging scale. The dependency on the merging scale in these distributions is greater than that seen in figure 2 which is to be expected since we are now correcting more emissions and therefore the mismatch between the parton shower and matrix element regions of phase space is greater.

Since we are now including higher multiplicity channels in our merging algorithm we check the distributions of scales at which higher numbers of jets are resolved. This is done in figure 7 for the resolution of four and five jets in the Durham and LUCLUS jet measures.

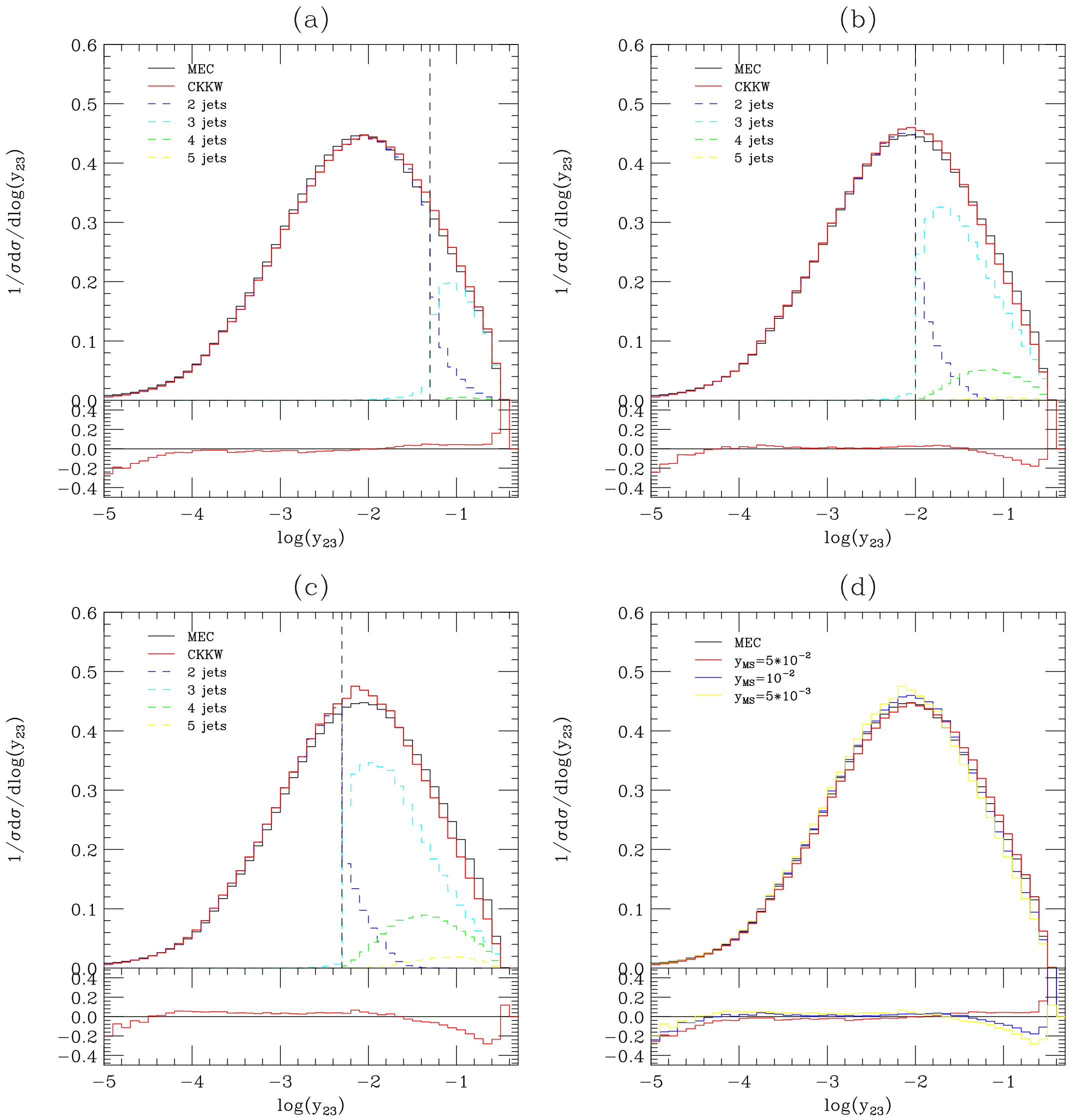

Figure 5. Distributions of the scale at which three jets are resolved in the Durham jet measure. The red line in plots (a)-(c) shows the distributions for the CKKW treatment with all multiplicity channels (up to a maximum of five jets) included at a set of merging scale choices in the Durham jet measure. Plot (d) gives a comparison of the different merging scale choices.

The merging in these distributions is well behaved.

These distributions demonstrate a degree of insensitivity to the choice of merging scale, which has been varied over an order of magnitude, however there is still some residual dependence on this choice. While the parton shower and merged matrix element treatments formally have the same large logarithm behavior, there are differences between the two. The degree of these differences will directly influence the amount of residual dependence on the merging scale that is observed. In changing the merging scale we are changing the volume of the matrix element phase space region and therefore changing the proportion of parton emissions that are corrected by exact matrix elements. Table 1 gives the cross

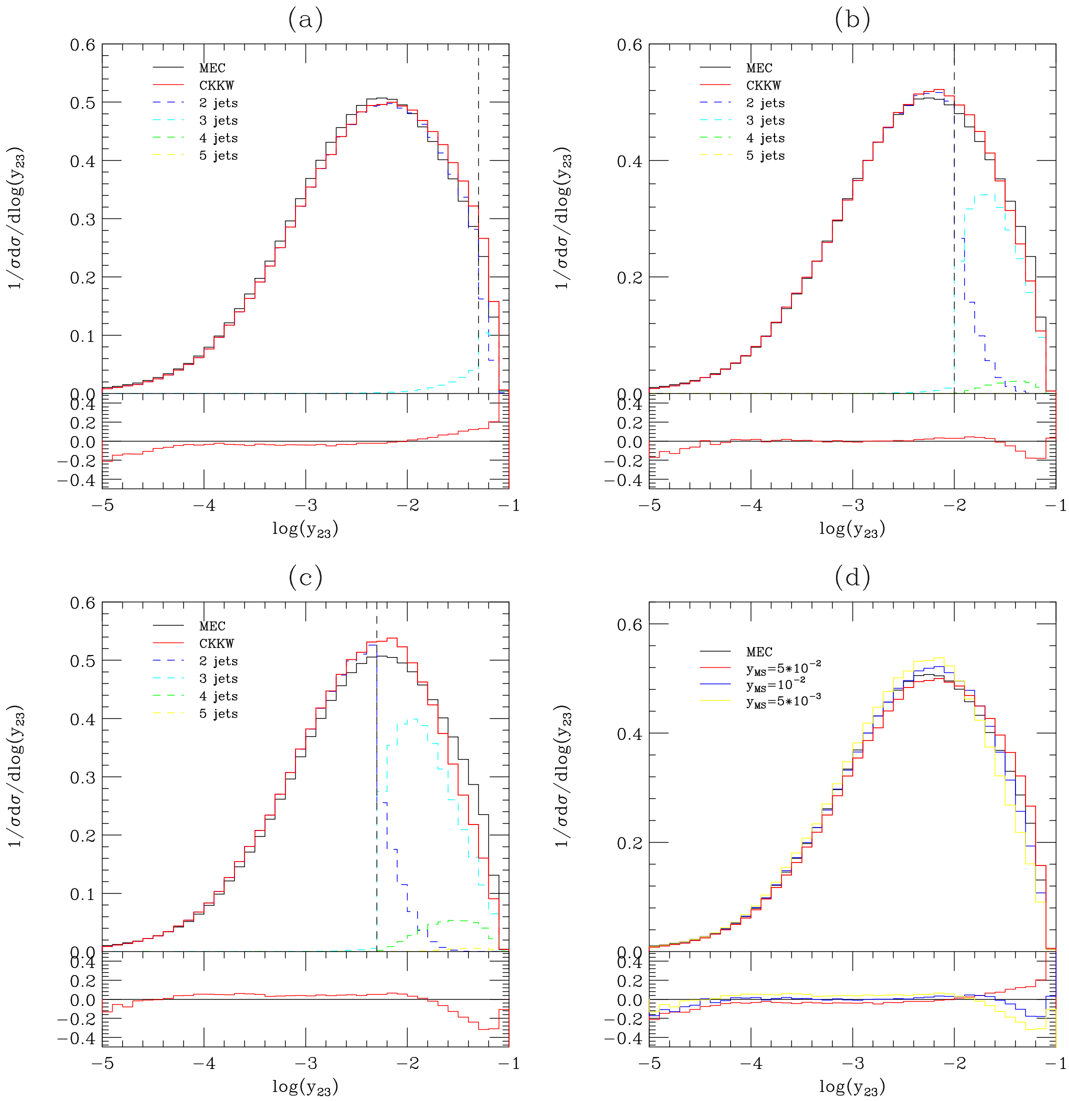

Figure 6. Distributions of the scale at which three jets are resolved in the LUCLUS jet measure. The red line in plots (a)-(c) shows the distributions for the CKKW treatment with all multiplicity channels (up to a maximum of five jets) included at a set of merging scale choices in the LUCLUS jet measure. Plot (d) gives a comparison of the different merging scale choices.

$y_{\rm MS}$	Durham cross section / nb	LUCLUS cross section / nb
5×10^{-2}	38.2	38.6
10^{-2}	36.5	37.1
5×10^{-3}	35.7	35.9

Table 1. Table of cross sections of the process $e^+e^- \to$ hadrons for different choices of the merging scale in the Durham and LUCLUS jet measures.

sections for the CKKW treatment at different choices of the merging scale and exhibits variation at the 5% level.

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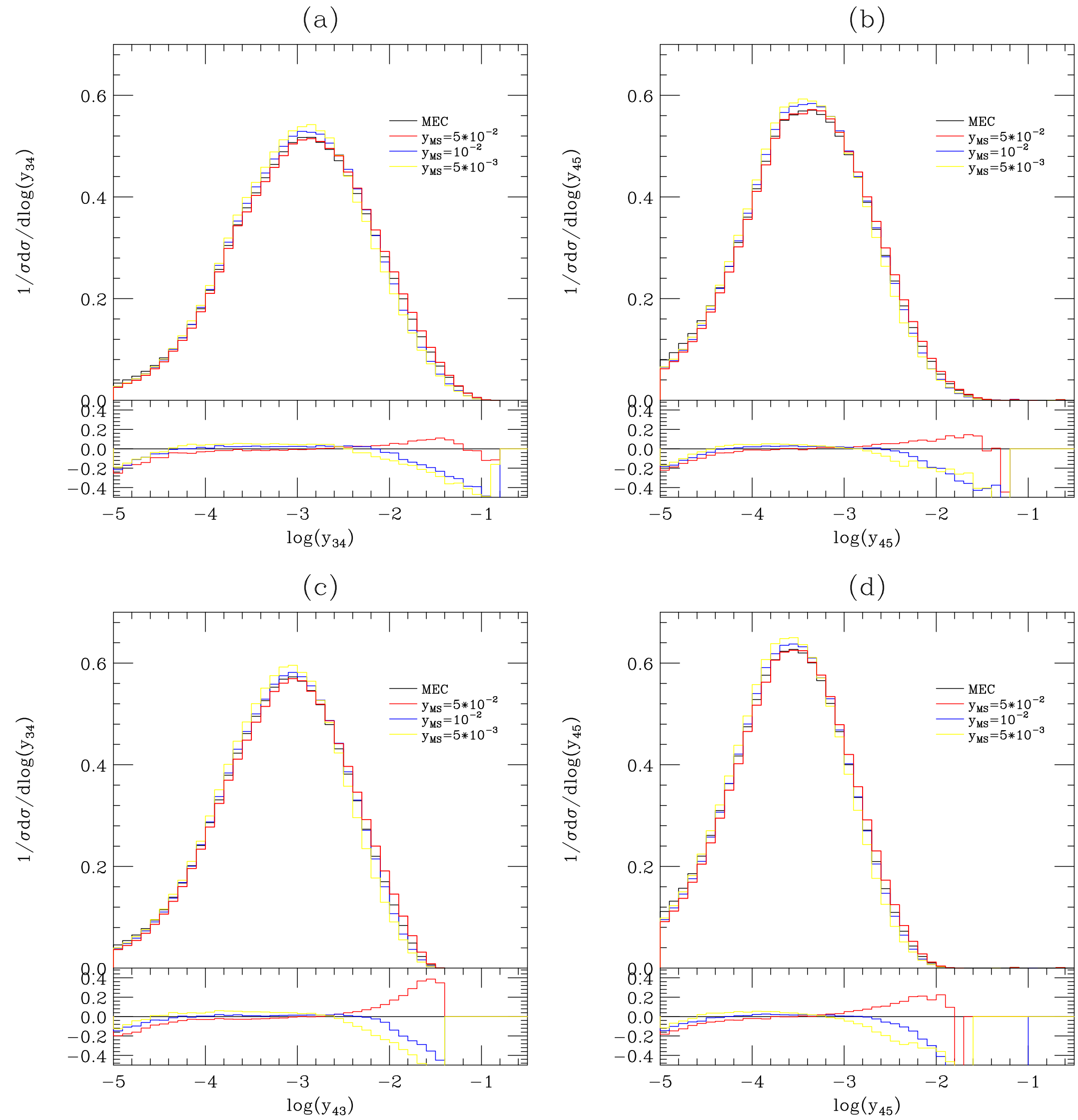

Figure 7. Distributions of the scale at which (a) four and (b) five jets are resolved in the Durham jet measure and the resolution scales for (c) four and (d) five jets in the LUCLUS jet measure.

5.2 Hadron level results

We present a comparison of the Herwig++ CKKW implementation with hadronization switched on to LEP data for a variety of event shapes. It is standard practice to *tune* the free parameters of an event generator to LEP data and this has been done with the default Herwig++ parton shower with matrix element corrections. Since the CKKW merging algorithm significantly changes the parton shower component of the event generator and in order to provide a fair comparison with default Herwig++, a new tune was performed on the free parameters for Herwig++ with the CKKW algorithm. This tune was performed with the merging scale set to $y_{\rm MS} = 10^{-2}$ in the Durham jet measure.

Figures 8–10 show distributions of a range of event shape, jet resolution and four-jet observables in comparison to LEP data. The parton level analysis shows that the merging scale choice of $y_{\rm MS} = 5\times10^{-2}$ leaves only a very small region of phase space that is corrected by the matrix elements. This very high scale choice will therefore not give the improvement expected in introducing the merging algorithm, we therefore omit this merging scale choice from the hadron level analysis. In each of the figures the red band shows the variation in distributions over the four merging scale choices of $y_{\rm MS} = 10^{-2}$ and $y_{\rm MS} = 5 \times 10^{-3}$ in the Durham and LUCLUS jet measures.

The CKKW distributions (red band) in figures 8–10 all demonstrate improved descriptions of the data in comparison to the default Herwig++ parton shower with matrix element corrections. In particular the tails of the distributions in figure 8, corresponding to hard emissions, and the jet resolution distributions of figure 9 with four and five jets are significantly improved as would be expected given the aims of the merging algorithm. The four-jet angle distributions of figure 10 are also all improved, with the exception of the α_{34} angle, which was already well described by the default Herwig++ parton shower. The $\theta_{\rm NR}$ distribution provides the most notable improvement in its description of the data in comparison to the default Herwig++ parton shower.

The width of the red band on the distributions shows that there is some residual dependence on the merging scale however it does not appear to be too serious and is at a similar level to that observed at parton level. This shows that the problems with colour structure, that appear in the standard CKKW algorithm, are not present here and that the truncated shower is working as intended. It should be noted that a fixed set of Herwig++ shower and hadronization parameters was used for each of the four merging scale choices; the variation would be reduced further if a tune of the parameters was performed for each merging scale choice.

The χ^2 per degree of freedom values for the distributions in figures 8–10 are given in table 2 for the merging scale choice of $y_{\rm MS} = 10^{-2}$ in the Durham jet measure, which was used in the tune. The CKKW values are lower than those of the default Herwig++ shower in all cases except for the α_{34} angle, where the default implementation already gave a satisfactory description, and in many cases the CKKW values are significantly lower.

6 Conclusions

A modified version of the CKKW algorithm has been implemented in Herwig++ for the process $e^+e^- \to$ hadrons. The modified algorithm uses truncated showers in order to provide smooth merging between the Herwig++ angular-ordered parton shower and a set of transverse-momentum-ordered emissions defined by inverting the Herwig++ momentum reconstruction procedure on a samples of parton momenta generated according to exact tree-level matrix elements.

The truncated shower was found to result in a smooth merging between parton shower and matrix element regions of phase space with parton level distributions appearing free of discontinuities around the merging scale and relatively insensitive to changes in the merging scale.

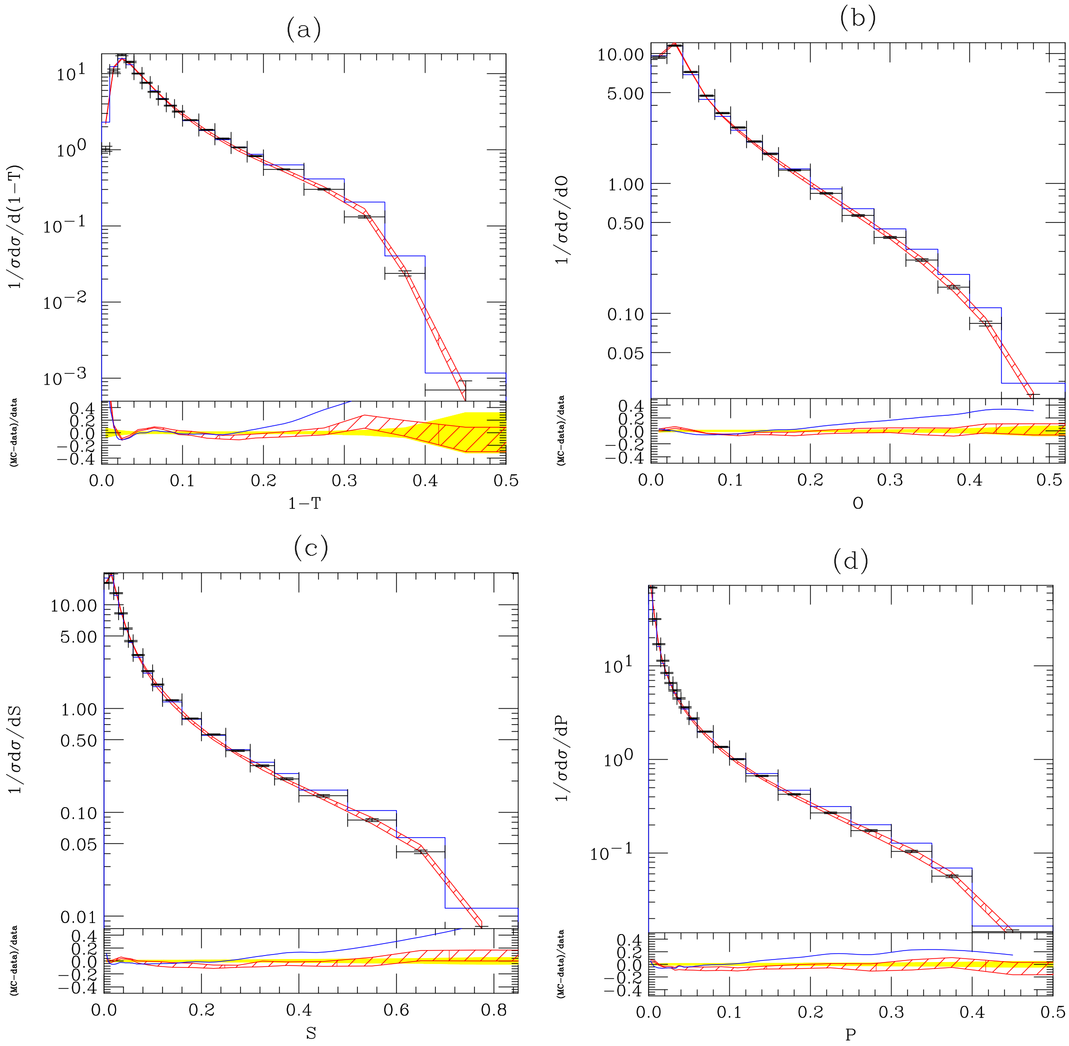

Figure 8. Distributions of the event shape variables (a) thrust, (b) oblateness, (c) sphericity and (d) planarity for $e^+e^- \to$ hadrons at a centre-of-mass energy of $\sqrt{s} = 91.2\,\mathrm{GeV}$ in comparison to LEP data (black) [32]. The red band gives the variation of the distributions of the CKKW implementation with merging scales choices of $y_{\rm MS} = 10^{-2}$ and $y_{\rm MS} = 5 \times 10^{-3}$ in the Durham and LUCLUS jet measures. The blue histogram gives the distributions of the default Herwig++ parton shower with matrix element corrections. The lower panel shows the ratio of the difference between simulation and data to the data in comparison to the error bounds of the data (yellow region).

A full tune of the Herwig++ free parameters was performed for the CKKW implementation with a merging scale of $y_{\rm MS} = 10^{-2}$ in the Durham jet measure. This was found to give a good description of LEP data, demonstrating a significant improvement over the results from the default Herwig++ parton shower with matrix element corrections applied.

The results show a comparable level of merging scale dependence and agreement with LEP data to that found in ref. [20], in which a similar CKKW merging approach was performed with a transverse-momentum-ordered dipole shower.

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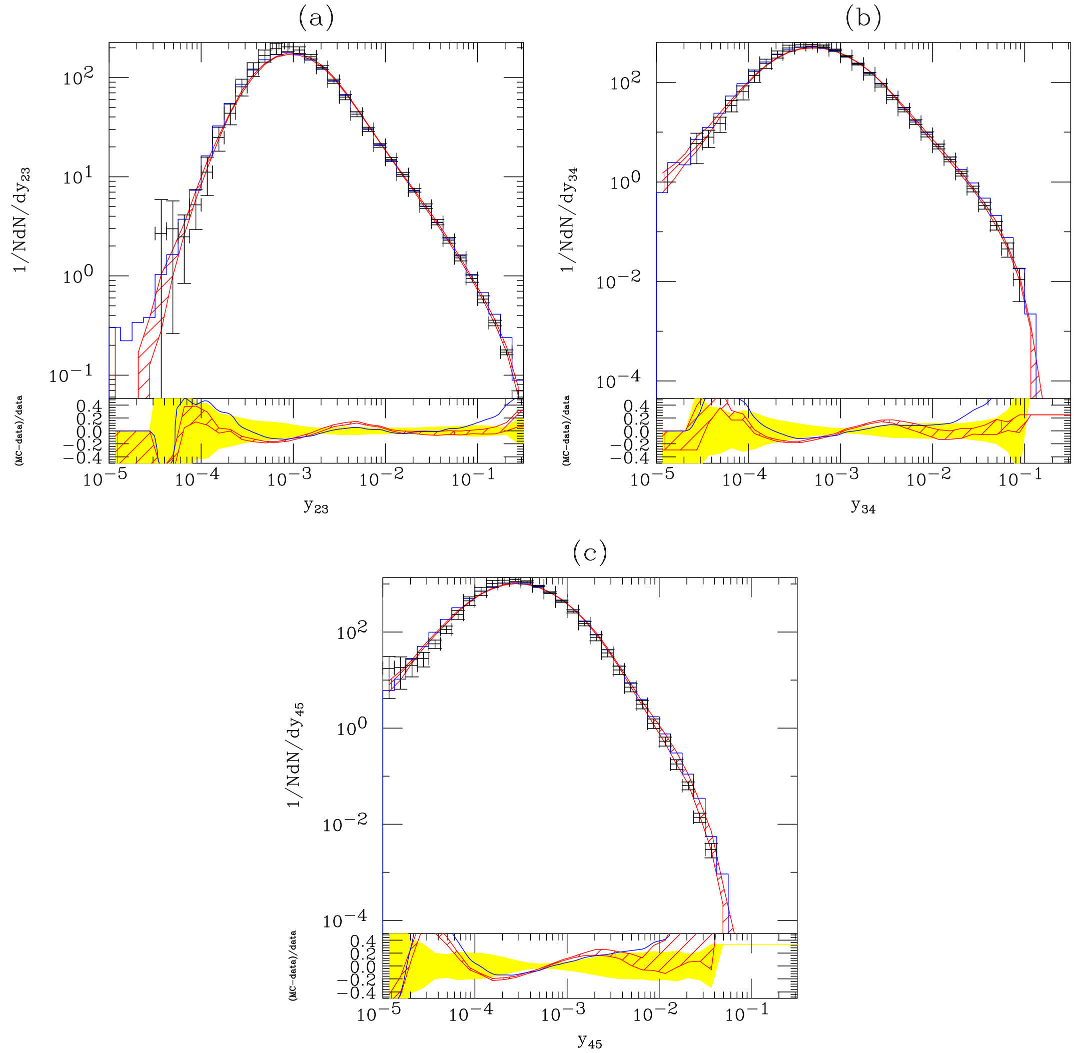

Figure 9. Distributions of the scale at which (a) three, (b) four and (c) five jets are resolved in the Durham jet measure for $e^+e^- \to$ hadrons at a centre-of-mass energy of $\sqrt{s} = 91.2\,\text{GeV}$ in comparison to LEP data [33]. The colours of the lines are the same as those in figure 8.

A Merging scale independence of the three-jet emission rate

In the following, we extend our pedagogical example concerning the merging of the two- and three-parton matrix element configurations to illustrate the cancellation of the merging scale dependence at NLL level.

For clarity of notation we define the remnant and vetoed Sudakov form factors

$$\Delta_i^{R,V}(\tilde{q}_1, \tilde{q}_2) = \frac{\Delta_i^{R,V}(\tilde{q}_1; k_{\perp_{\rm MS}})}{\Delta_i^{R,V}(\tilde{q}_2; k_{\perp_{\rm MS}})}, \tag{A.1}$$

where the dependence on the merging scale $k_{\perp_{\rm MS}}$ is implicit.

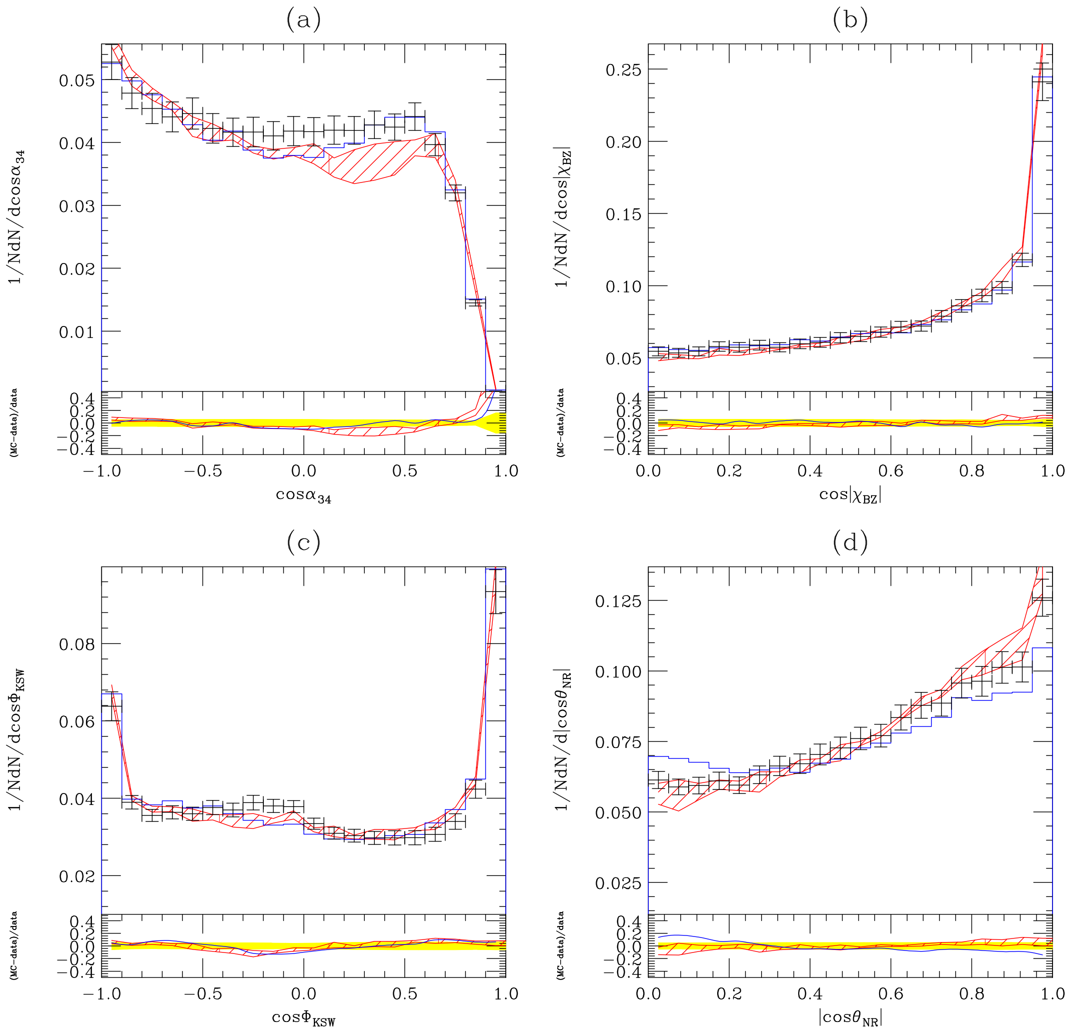

Figure 10. Distributions of four-jet angles for $e^+e^- \to$ hadrons at a centre-of-mass energy of $\sqrt{s} = 91.2\,\text{GeV}$ in comparison to LEP data [34]. figures (a)-(d) give the angle between the lowest energy jets α_{34}, the Bengtsson-Zerwas angle [29] χ_{BZ}, the Korner-Sielshotlz-Willrodt [31] Φ_{KSW} and the Nachtmann-Reiter angle [30] θ_{NR}. The colours of the lines are the same as those in figure 8.

The parton shower rate for the production of three-parton configurations via emission from the quark line (and none from the anti-quark) is given by

$$\begin{aligned}\mathcal{P}^{q\bar{q}g} &= \frac{1}{\sigma_{\text{tot}}}\,\mathrm{d}\sigma^{q\bar{q}}\,\mathrm{d}\mathcal{P}_{q\to qg}\left(\tilde{q},z\right) \\ &\qquad\qquad\times\Delta_q\left(\tilde{q}_I,\,\tilde{q}\right)\,\Delta_q\left(z\tilde{q},\,\tilde{q}_0\right)\,\Delta_g\left((1-z)\tilde{q},\,\tilde{q}_0\right)\,\Delta_{\bar{q}}\left(\tilde{q}_I,\,\tilde{q}_0\right).\end{aligned} \tag{A.2}$$

In the modified CKKW algorithm, emissions above the merging scale are generated according to the hard matrix element, initially with probability

$$\mathcal{P}^{q\bar{q}g}\left(y_{ij} > y_{\text{MS}}\right) = \frac{1}{\sigma_{\text{tot}}}\,\mathrm{d}\sigma^{q\bar{q}g}. \tag{A.3}$$

Depending on whether the p_T of the gluon is smaller with respect to the quark or anti-quark a shower history is assigned to the configuration in which the former or latter is deemed

Observable	Hw+ME χ^2/d.o.f	CKKW χ^2/d.o.f
Thrust	25.78	10.62
Sphericity	9.126	0.580
Oblateness	7.262	0.339
Planarity	3.928	1.211
y_{23}	2.812	0.867
y_{34}	1.912	1.026
y_{45}	4.204	2.018
$\cos\alpha_{34}$	1.043	3.301
$\cos\chi_{\mathrm{BZ}}$	0.3138	0.775
$\cos\Phi_{\mathrm{KSW}}$	1.645	1.337
$\cos\theta_{\mathrm{NR}}$	2.514	0.702

Table 2. A comparison of the χ^2 per degree of freedom for event shape observables in $e^+e^- \to$ hadrons with default Herwig++, with matrix element corrections, and the CKKW implementation, with merging scale set to $y_{\mathrm{MS}} = 10^{-2}$ in the Durham jet measure.

to have emitted the gluon. Let us assume that the gluon p_T with respect to the quark was the smaller of the two, the event is then assigned the relevant Sudakov and coupling constant weights

$$\mathcal{P}^{q\bar{q}g}\left(y_{ij} > y_{\mathrm{MS}}\right) \to \frac{1}{\sigma_{\mathrm{tot}}}\,\mathrm{d}\sigma^{q\bar{q}g}\,\frac{\alpha_S\left(p_\perp\left(\tilde{q},\,z\right)\right)}{\alpha_{S_{\mathrm{ME}}}} \\ \times\Delta_q^R\left(\tilde{q}_I,\,\tilde{q}\right)\,\Delta_q^R\left(z\tilde{q},\,\tilde{q}_0\right)\,\Delta_g^R\left(\left(1-z\right)\tilde{q},\,\tilde{q}_0\right)\,\Delta_{\bar{q}}^R\left(\tilde{q}_I,\,\tilde{q}_0\right). \tag{A.4}$$

For this configuration to remain a three-parton configuration, no further radiation should be generated in the truncated and vetoed shower. The probability of generating no emissions in the vetoed and truncated showers is found from eq. (3.9) and eq. (3.10) respectively. The result is that the emission probability receives further vetoed and remnant Sudakov form factors, resulting in the aggregate emission probability

$$\mathcal{P}^{q\bar{q}g}\left(y_{ij} > y_{\mathrm{MS}}\right) = \frac{1}{\sigma_{\mathrm{tot}}}\,\mathrm{d}\sigma^{q\bar{q}g}\,\frac{\alpha_S\left(p_\perp\left(\tilde{q},\,z\right)\right)}{\alpha_{S_{\mathrm{ME}}}} \\ \times\Delta_q^R\left(\tilde{q}_I,\,\tilde{q}\right)\,\Delta_q^R\left(z\tilde{q},\,\tilde{q}_0\right)\,\Delta_g^R\left(\left(1-z\right)\tilde{q},\,\tilde{q}_0\right)\,\Delta_{\bar{q}}^R\left(\tilde{q}_I,\,\tilde{q}_0\right) \\ \times\Delta_q^V\left(\tilde{q}_I,\,\tilde{q}\right)\,\Delta_q^V\left(z\tilde{q},\,\tilde{q}_0\right)\,\Delta_g^V\left(\left(1-z\right)\tilde{q},\,\tilde{q}_0\right)\,\Delta_{\bar{q}}^V\left(\tilde{q}_I,\,\tilde{q}_0\right)\,. \tag{A.5}$$

Recalling the definitions of the vetoed and remnant Sudakov form factors (eqs. (3.6) and (3.12)), it is clear from the fact that each remnant Sudakov form factor is accompanied by an analogous vetoed Sudakov form factor, that the emission rate does not depend on the merging scale y_{MS}.

Furthermore, if we take the NLL approximation of this we may replace $\mathrm{d}\sigma^{q\bar{q}g}$ with the factorized form it approaches in the soft and collinear limits,[13]

$$\mathrm{d}\sigma^{q\bar{q}g} \to \frac{\alpha_{S_{\mathrm{ME}}}}{\alpha_S\left(p_\perp\left(\tilde{q},\,z\right)\right)}\,\mathrm{d}\sigma^{q\bar{q}}\,\mathrm{d}\mathcal{P}_{q\to qg}\left(\tilde{q},z\right)\,. \tag{A.6}$$

[13] The ratio of coupling constants enters here due to the fact that $\mathrm{d}\sigma^{q\bar{q}g}$ contains the fixed coupling, while the shower branching probability contains the running coupling evaluated at $p_\perp$.

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In this approximation we see that the emission rate is identical to that of the parton shower in eq. (A.2),

$$\mathcal{P}^{q\bar{q}g}\left(y_{ij} > y_{\rm MS}\right) \approx \mathcal{P}^{q\bar{q}g}. \tag{A.7}$$

Beneath the merging scale three-parton configurations arise through the emission of a single parton from a configuration generated according to the two-parton matrix element. These hard two-parton configurations are initially generated with probability

$$\mathcal{P}^{q\bar{q}} = \frac{1}{\sigma_{\rm tot}}\, \mathrm{d}\sigma^{q\bar{q}} \tag{A.8}$$

and then reweighted according to the prescription in Sect 4, such that

$$\mathcal{P}^{q\bar{q}} \to \frac{1}{\sigma_{\rm tot}}\, \mathrm{d}\sigma^{q\bar{q}}\, \Delta_q^R\left(\tilde{q}_I,\, \tilde{q}_0\right)\, \Delta_{\bar{q}}^R\left(\tilde{q}_I,\, \tilde{q}_0\right)\, . \tag{A.9}$$

It follows from the vetoed shower equation (eq. (3.9)) that the aggregate probability for an emission to be subsequently generated from the quark line (and none from the external anti-quark line) is

$$\begin{aligned}\mathcal{P}^{q\bar{q}g}\left(y_{ij} < y_{\rm MS}\right) = \frac{1}{\sigma_{\rm tot}}\, &\mathrm{d}\sigma^{q\bar{q}}\, \mathrm{d}\mathcal{P}_{q\to qg}\left(\tilde{q}, z\right) \\ &\times\Delta_q^R\left(\tilde{q}_I,\, \tilde{q}\right)\, \Delta_q^R\left(\tilde{q},\, \tilde{q}_0\right)\, \Delta_{\bar{q}}^R\left(\tilde{q}_I,\, \tilde{q}_0\right) \\ &\times\Delta_q^V\left(\tilde{q}_I, \tilde{q}\right)\, \Delta_q^V\left(z\tilde{q}, \tilde{q}_0\right)\, \Delta_g^V\left(\left(1-z\right)\tilde{q}, \tilde{q}_0\right)\, \Delta_{\bar{q}}^V\left(\tilde{q}_I, \tilde{q}_0\right)\, .\end{aligned} \tag{A.10}$$

where, for comparison with eq. (A.6) we have rewritten the first remnant Sudakov form factor in eq. (A.10) as $\Delta_q^R\left(\tilde{q}_I,\, \tilde{q}\right)\, \Delta_q^R\left(\tilde{q},\, \tilde{q}_0\right)$. It appears that the remnant and vetoed Sudakov factors in eq. (A.6) do not match, spoiling the cancellation of the merging scale, however we note that we rectify this by writing one of the remnant Sudakovs as

$$\Delta_q^R\left(\tilde{q}, \tilde{q}_0\right) \approx \Delta_q^R\left(z\tilde{q}, \tilde{q}_0\right)\Delta_g^R\left(\left(1-z\right)\tilde{q}, \tilde{q}_0\right). \tag{A.11}$$

This replacement results in only subleading differences; we can see this by considering the soft and non-soft emission regions separately.[14] The region of soft emissions corresponds to the limit $z \to 1$ where $\Delta_g^R((1-z)\tilde{q}, \tilde{q}_0) \to 1$ and $\Delta_q^R(z\tilde{q}, \tilde{q}_0) \to \Delta_q^R(\tilde{q}, \tilde{q}_0)$, satisfying eq. (A.11). Away from the soft region we have $\tilde{q} \approx p_\perp$ and since $p_\perp < p_{\perp_{\rm MS}}$, the θ-function results in all remnant Sudakov factors approaching one, so eq. (A.11) is trivially satisfied. Making this approximation we find that, to NLL accuracy, the emission rate is independent of the merging scale and is given by the parton shower emission rate of eq. (A.2),

$$\mathcal{P}^{q\bar{q}g}\left(y_{ij} > y_{\rm MS}\right) \approx \mathcal{P}^{q\bar{q}g}. \tag{A.12}$$

We see that to NLL the proposed algorithm yields emission rates that are independent of the merging scale and are identical to the emission rates of the parton shower. We note that all three components, truncated showers, vetoed showers, Sudakov reweighting, are essential in achieving this smooth merging and independence from the merging scale.

[14] The arguments here are exactly those used in the POWHEG shower reorganisation.

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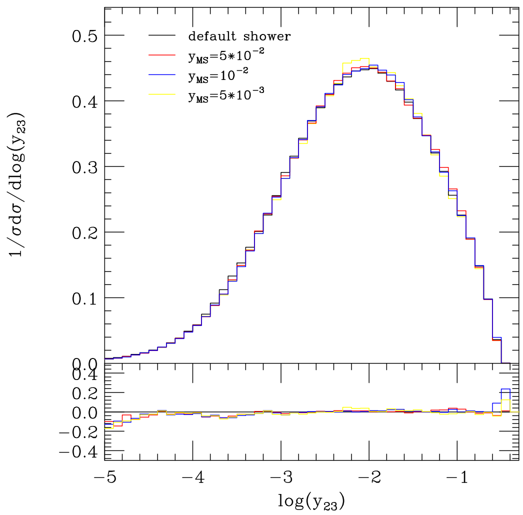

Figure 11. Parton level distributions of the scale at which three jets are resolved in the Durham jet measure for $e^+e^- \to$ hadrons at $\sqrt{s} = 91.2\,\text{GeV}$ comparing the default parton shower with no matrix element correction (black line) to a parton shower merged around y_{MS} with with merging scales set to $y_{\text{MS}} = 5 \times 10^{-2}$, $y_{\text{MS}} = 10^{-2}$ and $y_{\text{MS}} = 5 \times 10^{-3}$ in the Durham jet measure. The lower panel in each of the plots shows $(m - d)/d$ where m is the merged distribution and d is the default distribution.

B Parton shower merging test

As a check of the validity of the pseudo-shower history assignment procedure (as outlined in section 4.4) and the approximations made in applying the vetoes (as outlined in section 4.3), we present a test of reproducing the shower by merging two showers describing emissions above and below the merging scale y_{MS}.

The test was performed by generating the parton shower with a veto applied such that only events with $k_\perp > k_{\perp\text{MS}}$ are produced. The partons produced by this shower represent the shower approximation to a set of hard emissions. This set of hard emissions are then read back into the parton shower and showered with the CKKW truncated and vetoed showers, generating emissions with $k_\perp < k_{\perp\text{MS}}$.

Ideally the resultant distributions would exactly match those of the default shower. In practice, we have used approximations in the vetoes and an inexact history assignment and so we expect some differences. There are also some subleading differences inherent in the shower reorganisation as discussed in section 3.

Figure 11 shows parton level three-jet resolution distributions in the Durham jet measure which was also used to define the merging scale. We expect these plots to be sensitive to any problems that may arise. Figure 11 shows that the merged shower matches the default shower result closely for all three merging scale choices. As expected the distributions exhibit some slight differences however these are at an acceptable level, indicating that the approximations made in the vetoes and pseudo-shower history are valid.

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